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LOCAL WELL-POSEDNESS FOR THE (N+1)-DIMENSIONAL YANG-MILLS AND YANG-MILLS-HIGGS SYSTEM IN TEMPORAL GAUGE

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Abstract. The Yang-Mills and Yang-Mills-Higgs equations in temporal gauge are locally well-posed for small and rough initial data, which can be shown using the null structure of the critical bilinear terms. This carries over a similar result by Tao for the Yang-Mills equations in the (3+1)-dimensional case to the more general Yang-Mills-Higgs system and to general dimensions.

1. Introduction and main results

Let $\mathcal{G}$ be the Lie goup $SO(n,\mathbb{R})$ (the group of orthogonal matrices of determinant 1) or $SU(n,\mathbb{C})$ (the group of unitary matrices of determinant 1) and g its Lie algebra $so(n,\mathbb{R})$ (the algebra of trace-free skew symmetric matrices) or $su(n,\mathbb{C})$ (the algebra of trace-free skew hermitian matrices) with Lie bracket $[X,Y] = XY - YX$ (the matrix commutator). For given $A_\alpha : \mathbb{R}^{1+n} \to g$ we define the curvature by

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta]\,,$$

where $\alpha, \beta \in \{0,1,...,n\}$ and $D_\alpha = \partial_\alpha + [A_\alpha, \cdot\,]$.

Then the Yang-Mills system is given by

$$D^\alpha F_{\alpha\beta} = 0 \tag{1}$$

in Minkowski space $\mathbb{R}^{1+n} = \mathbb{R}_t \times \mathbb{R}^n_x$, where $n \ge 3$, with metric $diag(-1,1,...,1)$. Greek indices run over $\{0,1,...,n\}$, Latin indices over $\{1,...,n\}$, and the usual summation convention is used. We use the notation $\partial_\mu = \frac{\partial}{\partial x_\mu}$, where we write $(x^0, x^1, ..., x^n) = (t, x^1, ..., x^n)$ and also $\partial_0 = \partial_t$.

Setting $\beta = 0$ in (1) we obtain the Gauss-law constraint

$$\partial^j F_{j0} + [A^j, F_{j0}] = 0\,.$$

The system is gauge invariant. Given a sufficiently smooth function $U : \mathbb{R}^{1+n} \to \mathcal{G}$ we define the gauge transformation T by $TA_0 = A_0'$, $T(A_1,...,A_n) = (A_1',...,A_n')$, where

$$A_\alpha \longmapsto A_\alpha' = UA_\alpha U^{-1} - (\partial_\alpha U)U^{-1}\,.$$

It is well-known that if $(A_0, ...A_n)$ satisfies (1) so does $(A_0', ..., A_n')$.

Hence we may impose a gauge condition. We exlusively study the temporal gauge $A_0 = 0$.

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The Yang-Mills-Higgs system is given by

$$D^\alpha F_{\alpha\beta} = [D_\beta \phi, \phi] \tag{2}$$
$$D^\alpha D_\alpha \phi = |\phi|^{N-1}\phi\,. \tag{3}$$

Setting $\beta = 0$ in (2) we obtain the Gauss-law constraint

$$\partial^j F_{j0} = -[A^j, F_{j0}] + [D_0\phi, \phi]$$

where $\phi : \mathbb{R}^{1+n} \to g$. This system is also gauge invariant. Similarly as above we define the gauge transformation T by $TA_0 = A_0'$, $T(A_1, ..., A_n) = (A_1', ..., A_n')$, $T\phi = \phi'$, where

$$A_\alpha \longmapsto A_\alpha' = UA_\alpha U^{-1} - (\partial_\alpha U)U^{-1}$$
$$\phi \longmapsto \phi' = U\phi U^{-1}\,.$$

If $(A_0, ..., A_n, \phi)$ satisfies (2),(3), so does $(A_0', ..., A_n', \phi')$.

Some historical remarks: Concerning the well-posedness problem for the Yang-Mills equation in three space dimensions Klainerman and Machedon [KM1] proved global well-posedness in energy space in the temporal gauge. Selberg and Tesfahun [ST] proved local well-posedness for finite energy data in Lorenz gauge. This result was improved by Tesfahun [Te] to data without finite energy, namely for $(A(0), (\partial_t A)(0) \in H^s \times H^{s-1}$ with $s > \frac{6}{7}$. Local well-posedness in energy space was given by Oh [O] using a new gauge, namely the Yang-Mills heat flow. He was also able to shows that this solution can be globally extended [O1]. Tao [T1] showed local well-posedness for small data in $H^s \times H^{s-1}$ for $s > \frac{3}{4}$ in temporal gauge. In space dimension four where the energy space is critical with respect to scaling Klainerman and Tataru [KT] proved small data local well-posedness for a closely related model problem in Coulomb gauge for $s > 1$. Very recently this result was significantly improved by Krieger and Tataru [KrT], who were able to show global well-posedness for data with small energy. Sterbenz [St] considered also the four-dimensional case in Lorenz gauge and proved global well-posedness for small data in Besov space $\dot{B}^{1,1} \times \dot{(B)}^{0,1}$. In high space dimension $n \geq 6$ (and n even) Krieger and Sterbenz [KrSt] proved global well-posedness for small data in the critical Sobolev space.

Concerning the more general Yang-Mills-Higgs system Eardley and Moncrief [EM],[EM1] proved local and global well-posedness for initial data $(A(0), (\partial_t A)(0)$ and $(\phi(0), (\partial_t \phi)(0))$ in $H^s \times H^{s-1}$ and $s \geq 2$. In Coulomb gauge global well-posedness in energy space $H^1 \times L^2$ was shown by Keel [K]. Recently Tesfahun [Te1] considered the problem in Lorenz gauge and obtained local well-posedness in energy space.

We now study the Yang-Mills equation and also the Yang-Mills-Higgs system in arbitrary space dimension $n \geq 3$ in temporal gauge for low regularity data, which in three space dimension not necessarily have finite energy and which fulfill a smallness assumption, which reads in the Yang-Mills-Higgs case as follows

$$\|A(0)\|_{H^s} + \|(\partial_t A)(0)\|_{H^{s-1}} + \|\phi(0)\|_{H^s} + \|(\partial_t \phi)(0)\|_{H^{s-1}} < \epsilon$$

with a sufficiently small $\epsilon > 0$, under the assumption $s > \frac{3}{4}$ for $n = 3$ and $s > \frac{n}{2} - \frac{5}{8} - \frac{5}{8(2n-1)}$ in general dimension $n \geq 3$. We obtain a solution which satisfies $A, \phi \in C^0([0,1], H^s) \cap C^1([0,1], H^{s-1})$. A corresponding result holds for the Yang-Mills equation. Uniqueness holds in a certain subspace of Bourgain-Klainerman-Machedon type. The basis for our results is Tao's paper [T1]. We carry over his three-dimensional result for the Yang-Mills equation to the more general Yang-Mills-Higgs equations and to arbitrary dimensions $n \geq 3$. The result relies on the null structure of all the critical bilinear terms. We review this null

structure which was partly detected already by Klainerman-Machedon in the Yang-Mills case [KM1] and by Tesfahun [Te] for Yang-Mills-Higgs in the situation of the Lorenz gauge. The necessary estimates for the nonlinear terms in spaces of $X^{s,b}$-type in the (3+1)-dimensional case then reduce essentially to Tao's result [T1]. One of these estimates is responsible for the small data assumption. Because these local well-posedness results (Prop. 3.1) and (Prop. 3.2) can initially only be shown under the condition that the curl-free part A^{cf} of A (as defined below) vanishes for $t=0$ we have to show that this assumption can be removed by a suitable gauge transformation (Lemma 4.1) which preserves the regularity of the solution. This uses an idea of Keel and Tao [T1].

Our main results read as follows:

Theorem 1.1. *Let $n \ge 3$, $s > \frac{n}{2} - \frac{5}{8} - \frac{5}{8(2n-1)}$. Let $a \in H^s(\mathbb{R}^n)$, $a' \in H^{s-1}(\mathbb{R}^n)$ be given, where $a = (a_1,...,a_n)$, $a' = (a'_1,...,a'_n)$, satisfying the compatability condition $\partial^j a'_j = -[a^j, a'_j]$. Assume $\|a\|_{H^s} + \|a'\|_{H^{s-1}} \le \epsilon$, where $\epsilon > 0$ is sufficiently small. Then the Yang-Mills equation (1) in temporal gauge $A_0 = 0$ with initial conditions $A(0) = a$, $(\partial_t A)(0) = a'$, where $A = (A_1,...,A_n)$, has a unique local solution $A = A^{df}_+ + A^{df}_- + A^{cf}$, where*

$$A^{df}_\pm \in X^{s,\frac{3}{4}+}_\pm[0,1] \, , \, A^{cf} \in X^{s+\alpha,\frac{1}{2}+}_{\tau=0}[0,1] \, , \, \partial_t A^{cf} \in C^0([0,1], H^{s-1}) \, .$$

These spaces are defined below and $\alpha = \frac{3n+1}{8(2n-1)}$. This solution fulfills

$$A \in C^0([0,1], H^s(\mathbb{R}^n)) \cap C^1([0,1], H^{s-1}(\mathbb{R}^n)) \, .$$

Remark: In the (3+1)-dimensional case we assume $s > \frac{3}{4}$ and $\alpha = \frac{1}{4}$, so that data without finite energy are admissible. This is Tao's result [T1].

Theorem 1.2. *Let $n \ge 3$, $s > \frac{n}{2} - \frac{5}{8} - \frac{5}{8(2n-1)}$, and $2 \le N < 1 + \frac{7}{4(\frac{n}{2}-s)}$, if $s < \frac{n}{2}$, and $N < \infty$, if $s \ge \frac{n}{2}$. Here N is an odd integer, or $N \in \mathbb{N}$ with $N > s$. Let $a \in H^s(\mathbb{R}^n)$, $a' \in H^{s-1}(\mathbb{R}^n)$, $\phi_0 \in H^s(\mathbb{R}^3)$, $\phi_1 \in H^{s-1}(\mathbb{R}^3)$ be given, where $a = (a_1,...,a_n)$, $a' = (a'_1,...,a'_n)$, satisfying $\partial^j a'_j = -\partial^j a'_j - [\phi_1, \phi_0]$. Assume*

$$\|a\|_{H^s} + \|a'\|_{H^{s-1}} + \|\phi_0\|_{H^s} + \|\phi_1\|_{H^{s-1}} \le \epsilon \, ,$$

where $\epsilon > 0$ is sufficiently small. Then the Yang-Mills-Higgs equations (2) , (3) in temporal gauge $A_0 = 0$ with initial conditions

$$A(0) = a \, , \, (\partial_t A)(0) = a' \, , \, \phi(0) = \phi_0 \, , \, (\partial_t \phi)(0) = \phi_1 \, ,$$

where $A = (A_1,...,A_n)$, has a unique local solution $A = A^{df}_+ + A^{df}_- + A^{cf}$ and $\phi = \phi_+ + \phi_-$, where

$$A^{df}_\pm \in X^{s,\frac{3}{4}+}_\pm[0,1], A^{cf} \in X^{s+\alpha,\frac{1}{2}+}_{\tau=0}[0,1], \partial_t A^{cf} \in C^0([0,1], H^{s-1}), \phi_\pm \in X^{s,\frac{3}{4}+}_\pm[0,1] \, ,$$

where these spaces are defined below and $\alpha = \frac{3n+1}{8(2n-1)}$. This solution fulfills

$$A \, , \, \phi \in C^0([0,1], H^s(\mathbb{R}^n)) \cap C^1([0,1], H^{s-1}(\mathbb{R}^n)) \, .$$

Remark: The assumption $N > s$ or N odd ensures that the function $f(s) = |s|^{N-1} s$ for $s \in \mathbb{R}$ is smooth enough at the origin.

We denote the Fourier transform with respect to space and time and with respect to space by $\widehat{\ }$ and $\mathcal{F}$, respectively. The operator $|\nabla|^\alpha$ is defined by $(\mathcal{F}(|\nabla|^\alpha f))(\xi) = |\xi|^\alpha (\mathcal{F}f)(\xi)$ and similarly $\langle \nabla \rangle^\alpha$. $\square = \partial_t^2 - \Delta$ is the d'Alembert operator.
$a+ := a + \epsilon$ for a sufficiently small $\epsilon > 0$, so that $a < a+ < a++$, and similarly $a-- < a- < a$, and $\langle \cdot \rangle := (1 + |\cdot|^2)^{\frac{1}{2}}$.

The standard spaces $X^{s,b}_{\pm}$ of Bourgain-Klainerman-Machedon type belonging to the half waves are the completion of the Schwarz space $\mathcal{S}(\mathbb{R}^4)$ with respect to the norm

$$\|u\|_{X^{s,b}_{\pm}} = \|\langle \xi \rangle^s \langle \tau \mp |\xi| \rangle^b \widehat{u}(\tau,\xi)\|_{L^2_{\tau\xi}} .$$

Similarly we define the wave-Sobolev spaces $X^{s,b}_{|\tau|=|\xi|}$ with norm

$$\|u\|_{X^{s,b}_{|\tau|=|\xi|}} = \|\langle \xi \rangle^s \langle |\tau| - |\xi| \rangle^b \widehat{u}(\tau,\xi)\|_{L^2_{\tau\xi}}$$

and also $X^{s,b}_{\tau=0}$ with norm

$$\|u\|_{X^{s,b}_{\tau=0}} = \|\langle \xi \rangle^s \langle \tau \rangle^b \widehat{u}(\tau,\xi)\|_{L^2_{\tau\xi}} .$$

We also define $X^{s,b}_{\pm}[0,T]$ as the space of the restrictions of functions in $X^{s,b}_{\pm}$ to $[0,T] \times \mathbb{R}^3$ and similarly $X^{s,b}_{|\tau|=|\xi|}[0,T]$ and $X^{s,b}_{\tau=0}[0,T]$. We frequently use the estimates $\|u\|_{X^{s,b}_{\pm}} \le \|u\|_{X^{s,b}_{|\tau|=|\xi|}}$ for $b \le 0$ and the reverse estimate for $b \ge 0$.

2. Reformulation of the problem and null structure

In temporal gauge $A_0 = 0$ the system (1) is equivalent to

$$\begin{aligned} \partial_t \, div \, A &= -[A_i, \partial_t A^i] \\ \Box A_j &= \partial_j \, div A - [div A, A_j] - 2[A^i, \partial_i A_j] + [A^i, \partial_j A_i] - [A^i, [A_i, A_j]] \end{aligned}$$

and the Gauss constraint reduces to

$$\partial^j \partial_t A_j = -[A_j, \partial_t A^j] .$$

Similarly in temporal gauge $A_0 = 0$ the system (2),(3) is equivalent to

$$\begin{aligned} \partial_t \, div \, A &= -[\partial_t \phi, \phi] - [A_i, \partial_t A^i] \\ \Box A_j &= \partial_j \, div A - [div A, A_j] - 2[A^i, \partial_i A_j] + [A^i, \partial_j A_i] - [\phi, \partial_j \phi] - [A^i, [A_i, A_j]] \\ &\quad - [\phi, [A_j, \phi]] \\ \Box \phi &= -[div A, \phi] - 2[A_i, \partial^i \phi] - [A^i, [A_i, \phi]] + |\phi|^{N-1} \phi \end{aligned}$$

and the Gauss constraint reduces to

$$\partial^j \partial_t A_j = -[A_j, \partial_t A^j] - [\partial_t \phi, \phi] .$$

We decompose A into its divergence-free part A^{df} and its curl-free part A^{cf} :

$$A = A^{df} + A^{cf} ,$$

where

$$A^{df}_j = (PA)_j := R^k(R_j A_k - R_k A_j) \quad , \quad A^{cf}_j = -R_j R_k A^k .$$

Here P denotes the Leray projection onto the divergence-free part, and $R_j := |\nabla|^{-1} \partial_j$ is the Riesz transform.

Then we obtain the following system which is equivalent to (1):

$$\partial_t A^{cf} = (-\Delta)^{-1} \nabla [A_i, \partial_t A^i] \tag{4}$$

$$\Box A^{df} = -P[div \, A^{cf}, A] - 2P[A^i, \partial_i A] + P[A^i, \nabla A_i] - P[A^i, [A_i, A]] \tag{5}$$

Similarly the following system is equivalent to (2),(3):

$$\partial_t A^{cf} = -(-\Delta)^{-1} \nabla [\partial_t \phi, \phi] + (-\Delta)^{-1} \nabla [A_i, \partial_t A^i] \tag{6}$$

$$\begin{aligned} \Box A^{df} &= -P[div \, A^{cf}, A] - 2P[A^i, \partial_i A] + P[A^i, \nabla A_i] - P[\phi, \nabla \phi] - P[A^i, [A_i, A]] \\ &\quad - P[\phi, [A, \phi]] \end{aligned} \tag{7}$$

$$\Box \phi = -[div \, A^{cf}, \phi] - 2[A_i, \partial^i \phi] - [A^i, [A_i, \phi]] + |\phi|^{N-1} \phi . \tag{8}$$

We now show that all the critical terms in (5), (7) and (8), namely the quadratic terms which contain only A^{df} or ϕ have null structure. Those quadratic terms which contain A^{cf} are less critical, because A^{cf} is shown to be more regular than A^{df}, and the cubic terms are also less critical, because they contain no derivatives. The only critical term in (8) is $[A_i^{df}, \partial^i \phi]$. We easily calculate

$$\begin{aligned}
&[A_i^{df}, \partial^i A^{df}] = [R^k(R_i A_k - R_k A_i), \partial^i A^{df}] \\
&= \frac{1}{2}\big([R^k(R_i A_k - R_k A_i), \partial^i A^{df}] + [R^i(R_k A_i - R_i A_k), \partial^k A^{df}]\big) \\
&= \frac{1}{2}\big([R^k(R_i A_k - R_k A_i), \partial^i A^{df}] - [R^i(R_i A_k - R_k A_i), \partial^k A^{df}]\big) \\
&= \frac{1}{2} Q^{ik}[|\nabla|^{-1}(R_i A_k - R_k A_i), A^{df}] \qquad (9)
\end{aligned}$$

where

$$Q_{ij}[u,v] := [\partial_i u, \partial_j v] - [\partial_j u, \partial_i v] = Q_{ij}(u,v) + Q_{ji}(v,u)$$

with the standard null form

$$Q_{ij}(u,v) := \partial_i u \partial_j v - \partial_j u \partial_i v \,.$$

Thus, ignoring P, which is a bounded operator we obtain

$$P[A_i^{df}, \partial^i A^{df}] \sim \sum Q_{ik}[|\nabla|^{-1} A^{df}, A^{df}]\,, \qquad (10)$$

and similarly

$$P[A_i^{df}, \partial^i \phi] \sim \sum Q_{ik}[|\nabla|^{-1} A^{df}, \phi]\,. \qquad (11)$$

Moreover

$$\begin{aligned}
(\phi \nabla \phi')_j^{df} &= R^k(R_j(\phi \partial_k \phi') - R_k(\phi \partial_j \phi')) \\
&= |\nabla|^{-2} \partial^k(\partial_j(\phi \partial_k \phi') - \partial_k(\phi \partial_j \phi'))) \\
&= |\nabla|^{-2} \partial^k(\partial_j \phi \partial_k \phi' - \partial_k \phi \partial_j \phi') \\
&= |\nabla|^{-2} \partial^k Q_{jk}(\phi, \phi')
\end{aligned}$$

so that

$$P[\phi, \nabla \phi] \sim \sum |\nabla|^{-1} Q_{jk}[\phi, \phi]\,, \qquad (12)$$

and

$$P[A_i^{df}, \nabla A_i^{df}] \sim \sum |\nabla|^{-1} Q_{jk}[A^{df}, A^{df}]\,, \qquad (13)$$

All the other quadratic terms contain at least one factor A^{cf}.

Defining

$$\phi_\pm = \frac{1}{2}(\phi \mp i \langle \nabla \rangle^{-1} \partial_t \phi) \Longleftrightarrow \phi = \phi_+ + \phi_- \,,\; \partial_t \phi = i \langle \nabla \rangle (\phi_+ - \phi_-)$$

$$A_\pm^{df} = \frac{1}{2}(A^{df} \mp i \langle \nabla \rangle^{-1} \partial_t A^{df}) \Longleftrightarrow A^{df} = A_+^{df} + A_-^{df} \,,\; \partial_t A^{df} = i \langle \nabla \rangle (A_+^{df} - A_-^{df})$$

we can rewrite (4),(5) as

$$\partial_t A^{cf} = (-\Delta)^{-1} \nabla [A_i, \partial_t A^i] \qquad (14)$$

$$(i\partial_t \pm \langle \nabla \rangle) A_\pm^{df} = \mp 2^{-1} \langle \nabla \rangle^{-1} (R.H.S.\, of\, (5) - A^{df})\,. \qquad (15)$$

with initial data

$$A_\pm^{df}(0) = \frac{1}{2}(A^{df}(0) \mp i^{-1} \langle \nabla \rangle^{-1} \partial_t A^{df}(0)\,. \qquad (16)$$

Similarly we can rewrite (6),(7),(8) as

$$\partial_t A^{cf} = (-\Delta)^{-1}\nabla[\partial_t\phi,\phi] + (-\Delta)^{-1}\nabla[A_i,\partial_t A^i] \tag{17}$$

$$(i\partial_t \mp \langle\nabla\rangle)A^{df}_{\pm} = \mp 2^{-1}\langle\nabla\rangle^{-1}(R.H.S.\,of\,(7) - A^{df}) \tag{18}$$

$$(i\partial_t \mp \langle\nabla\rangle)\phi_{\pm} = \mp 2^{-1}\langle\nabla\rangle^{-1}(R.H.S.\,of\,(8) - \phi)\,. \tag{19}$$

The initial data are transformed as follows:

$$\phi_{\pm}(0) = \frac{1}{2}(\phi(0) \mp i^{-1}\langle\nabla\rangle^{-1}\partial_t\phi(0)) \tag{20}$$

$$A^{df}_{\pm}(0) = \frac{1}{2}(A^{df}(0) \mp i^{-1}\langle\nabla\rangle^{-1}\partial_t A^{df}(0)\,. \tag{21}$$

3. The preliminary local well-posedness results

We now state and prove preliminary local well-posedness of (4),(5) as well as (6),(7),(8), for which it is essential to have data for A with vanishing curl-free part.

Proposition 3.1. *For space dimension $n \ge 3$ assume $s > \frac{n}{2} - \frac{5}{8} - \frac{5}{8(2n-1)}$ and $\alpha = \frac{3n+1}{8(2n-1)}$. Let $a^{df} = (a^{df}_1,...,a^{df}_n) \in H^s$, $a'^{df} = (a'^{df}_1,...,a'^{df}_n) \in H^{s-1}$ be given with*

$$\sum_j \|a^{df}_j\|_{H^s} + \sum_j \|a'^{df}_j\|_{H^{s-1}} \le \epsilon_0\,,$$

where $\epsilon_0 > 0$ is sufficiently small. Then the system (4),(5) with initial conditions

$$A^{df}(0) = a^{df}\,,\ (\partial_t A^{df})(0) = a'^{df}\,,\ A^{cf}(0) = 0\,,$$

has a unique local solution

$$A = A^{df}_+ + A^{df}_- + A^{cf}\,,$$

where

$$A^{df}_{\pm} \in X^{s,\frac{3}{4}+}_{\pm}[0,1]\,,\ A^{cf} \in X^{s+\alpha,\frac{1}{2}+}_{\tau=0}[0,1]\,,\ \partial_t A^{cf} \in C^0([0,1],H^{s-1})\,.$$

Uniqueness holds (of course) for not necessarily vanishing initial data $A^{cf}(0) = a^{cf}$. The solution satisfies

$$A \in C^0([0,1],H^s) \cap C^1([0,1],H^{s-1})\,.$$

Proposition 3.2. *For space dimension $n \ge 3$ assume $s > \frac{n}{2} - \frac{5}{8} - \frac{5}{8(2n-1)}$ and $\alpha = \frac{3n+1}{8(2n-1)}$. Assume $2 \le N < 1 + \frac{7}{4(\frac{n}{2}-s)}$, if $s < \frac{n}{2}$, and $2 \le N < \infty$, if $s \ge \frac{n}{2}$. Here N is an odd integer, or $N \in \mathbb{N}$ with $N > s$. Let $a^{df} = (a^{df}_1,...,a^{df}_n) \in H^s$, $a'^{df} = (a'^{df}_1,...,a'^{df}_n) \in H^{s-1}$, $\phi_0 \in H^s$, $\phi_1 \in H^{s-1}$ be given with*

$$\sum_j \|a^{df}_j\|_{H^s} + \sum_j \|a'^{df}_j\|_{H^{s-1}} + \|\phi_0\|_{H^s} + \|\phi_1\|_{H^{s-1}} \le \epsilon_0\,,$$

where $\epsilon_0 > 0$ is sufficiently small. Then the system (6),(7),(8) with initial conditions

$$\phi(0) = \phi_0\,,\ (\partial_t\phi)(0) = \phi_1\,,\ A^{df}(0) = a^{df}\,,\ (\partial_t A^{df})(0) = a'^{df}\,,\ A^{cf}(0) = 0\,,$$

has a unique local solution

$$\phi = \phi_+ + \phi_- \quad , \quad A = A^{df}_+ + A^{df}_- + A^{cf}\,,$$

where

$$\phi_{\pm} \in X^{s,\frac{3}{4}+}_{\pm}[0,1], A^{df}_{\pm} \in X^{s,\frac{3}{4}+}_{\pm}[0,1], A^{cf} \in X^{s+\alpha,\frac{1}{2}+}_{\tau=0}[0,1], \partial_t A^{cf} \in C^0([0,1],H^{s-1})\,.$$

Uniqueness holds (of course) for not necessarily vanishing initial data $A^{cf}(0) = a^{cf}$. The solution satisfies

$$A, \phi \in C^0([0,1], H^s) \cap C^1([0,1], H^{s-1}) .$$

Fundamental for their proof are the following estimates.

Proposition 3.3. *Let $n \geq 2$.*

(1) *For $2 < q \leq \infty$, $2 \leq r < \infty$, $\frac{2}{q} = (n-1)(\frac{1}{2} - \frac{1}{r})$, $\mu = n(\frac{1}{2} - \frac{1}{r}) - \frac{1}{q}$ the following estimate holds*

$$\|u\|_{L_t^q L_x^r} \lesssim \|u\|_{X^{\mu,\frac{1}{2}+}_{|\tau|=|\xi|}} . \tag{22}$$

(2) *For $k \geq 0$, $p < \infty$ and $\frac{n-1}{2(n+1)} \geq \frac{1}{p} \geq \frac{n-1}{2(n+1)} - \frac{k}{n}$ the following estimate holds:*

$$\|u\|_{L_x^p L_t^2} \lesssim \|u\|_{X^{k+\frac{n-1}{2(n+1)},\frac{1}{2}+}_{|\tau|=|\xi|}} . \tag{23}$$

Proof. (22) is the Strichartz type estimate, which can be found for e.g. in [GV], Prop. 2.1, combined with the transfer principle.

Concerning (23) we use [KMBT], Thm. B.2:

$$\|\mathcal{F}_t u\|_{L^2_\tau L_x^{\frac{2(n+1)}{n-1}}} \lesssim \|u_0\|_{\dot{H}^{\frac{n-1}{2(n+1)}}} ,$$

if $u = e^{it|\nabla|}u_0$ and $\mathcal{F}_t$ denotes the Fourier transform with respect to time. This immediately implies by Plancherel, Minkowski's inequality and Sobolev's embedding theorem

$$\|u\|_{L_x^p L_t^2} = \|\mathcal{F}_t u\|_{L_x^p L_\tau^2} \leq \|\mathcal{F}_t u\|_{L_\tau^2 L_x^p} \lesssim \|\mathcal{F}_t u\|_{L^2_\tau H_x^{k,\frac{2(n+1)}{n-1}}} \lesssim \|u_0\|_{H^{k+\frac{n-1}{2(n+1)}}} .$$

The transfer principle implies (23). $\square$

Proof of Prop. 3.2 and Prop. 3.1. We use the system (14),(15) (instead of (4),(5)) and (17),(18),(19) (instead of (6),(7),(8)) with initial conditions (16) and (20),(21). We want to use a contraction argument for $A_\pm^{df} \in X_\pm^{s,\frac{3}{4}+\epsilon}[0,1]$, $A^{cf} \in X_{\tau=0}^{s+\alpha,\frac{1}{2}+\epsilon}[0,1]$, $\partial_t A^{cf} \in C^0([0,1], H^{s-1}\ )$, and in the Yang-Mills-Higgs case in addition for $\phi \in X_\pm^{s,\frac{3}{4}+\epsilon}[0,1]$. Provided that our small data assumption holds this can be reduced by well-known arguments to suitable multilinear estimates of the right hand sides of these equations. For (15) e.g. we make use of the following well-known estimate:

$$\|A_\pm^{df}\|_{X_\pm^{l,b}[0,1]} \lesssim \|A_\pm^{df}(0)\|_{H^l} + \|R.H.S.\, of\ (15)\|_{X_\pm^{l,b-1}[0,1]} ,$$

which holds for $l \in \mathbb{R}$, $\frac{1}{2} < b \leq 1$.

Thus the local existence and uniqueness can be reduced to the following estimates.

In order to control A^{cf} we need

$$\||\nabla|^{-1}(\phi_1 \partial_t \phi_2)\|_{X_{\tau=0}^{s+\alpha,-\frac{1}{2}+\epsilon+}} \lesssim \|\phi_1\|_{X^{s,\frac{3}{4}+\epsilon}_{|\tau|=|\xi|}} \|\phi_2\|_{X^{s,\frac{3}{4}+\epsilon}_{|\tau|=|\xi|}} \tag{24}$$

$$\||\nabla|^{-1}(\phi_1 \partial_t \phi_2)\|_{X_{\tau=0}^{s+\alpha,-\frac{1}{2}+2\epsilon-}} \lesssim \|\phi_1\|_{X_{\tau=0}^{s+\alpha,\frac{1}{2}+\epsilon}} \|\phi_2\|_{X_{\tau=0}^{s+\alpha,\frac{1}{2}+\epsilon}} \tag{25}$$

$$\begin{aligned} \||\nabla|^{-1}(\phi_1 \partial_t \phi_2)\|_{X_{\tau=0}^{s+\alpha,-\frac{1}{2}+\epsilon}} &+ \||\nabla|^{-1}(\phi_2 \partial_t \phi_1)\|_{X_{\tau=0}^{s+\alpha,-\frac{1}{2}+\epsilon}} \\ &\lesssim \|\phi_1\|_{X_{\tau=0}^{s+\alpha,\frac{1}{2}+\epsilon}} \|\phi_2\|_{X^{s,\frac{3}{4}+\epsilon}_{|\tau|=|\xi|}} . \end{aligned} \tag{26}$$

In order to control $\partial_t A^{cf}$ we need

$$\||\nabla|^{-1}(A_1\partial_t A_2)\|_{C^0(H^{s-1})} \lesssim (\|A_1^{cf}\|_{X^{s+\alpha,\frac{1}{2}+}_{\tau=0}} + \sum_{\pm} \|A^{df}_{1\pm}\|_{X^{s,\frac{1}{2}+}_{\pm}}) \\ (\|\partial_t A_2^{cf}\|_{C^0(H^{s-1})} + \sum_{\pm} \|A^{df}_{2\pm}\|_{X^{s,\frac{1}{2}+}_{\pm}}) . \tag{27}$$

The estimate for A^{df} and ϕ by use of (10),(11),(12),(13) reduces to

$$\|Q_{ij}(|\nabla|^{-1}\phi_1,\phi_2)\|_{X^{s-1,-\frac{1}{4}+2\epsilon}_{|\tau|=|\xi|}} + \|\nabla^{-1}Q_{ij}(\phi_1,\phi_2)\|_{X^{s-1,-\frac{1}{4}+2\epsilon}_{|\tau|=|\xi|}} \\ \lesssim \|\phi_1\|_{X^{s,\frac{3}{4}+\epsilon}_{|\tau|=|\xi|}} \|\phi_2\|_{X^{s,\frac{3}{4}+\epsilon}_{|\tau|=|\xi|}} . \tag{28}$$

For the proof of (28) we refer to [T], Prop. 9.2 (slightly modified), which is given under the assumption $s > \frac{n}{2} - \frac{3}{4}$. This assumption is weaker than our assumption, if $n \geq 4$, and they coincide for $n = 3$.

Moreover for the terms $P[div\, A^{cf}, A]$, $P[A^i, \partial_i A]$ and $P[A^i, \partial_j A_i]$ we need

$$\|\nabla A^{cf} A^{df}\|_{X^{s-1,-\frac{1}{4}+2\epsilon}_{|\tau|=|\xi|}} + \|A^{cf}\nabla A^{df}\|_{X^{s-1,-\frac{1}{4}+2\epsilon}_{|\tau|=|\xi|}} \lesssim \|A^{cf}\|_{X^{s+\alpha,\frac{1}{2}+\epsilon}_{\tau=0}} \|A^{df}\|_{X^{s,\frac{3}{4}+\epsilon}_{|\tau|=|\xi|}} \tag{29}$$

and

$$\|\nabla A^{cf} A^{cf}\|_{X^{s-1,-\frac{1}{4}+2\epsilon}_{|\tau|=|\xi|}} \lesssim \|A^{cf}\|^2_{X^{s+\alpha,\frac{1}{2}+\epsilon}_{\tau=0}} . \tag{30}$$

All the cubic terms are estimated by

$$\|A_1A_2A_3\|_{X^{s-1,-\frac{1}{4}+2\epsilon}_{|\tau|=|\xi|}} \lesssim \prod_{i=1}^{3} \min(\|A_i\|_{X^{s,\frac{3}{4}+\epsilon}_{|\tau|=|\xi|}}, \|A_i\|_{X^{s+\alpha,\frac{1}{2}+\epsilon}_{\tau=0}}) . \tag{31}$$

Remark that in (27), (29) and (31) A^{df} may be replaced by ϕ .
For the Yang-Mills-Higgs system we additionally need

$$\||\phi|^{N-1}\phi\|_{X^{s-1,-\frac{1}{4}+2\epsilon}_{|\tau|=|\xi|}} \lesssim \|\phi\|^N_{X^{s,\frac{3}{4}+\epsilon}_{|\tau|=|\xi|}} . \tag{32}$$

All these estimates up to (27) and (32) have been essentially given by Tao [T1] for the Yang-Mills case in space dimension $n = 3$. We remark that it is especially (26) which prevents a large data result, because it seems to be difficult to replace $X^{s+\alpha,-\frac{1}{2}+\epsilon}_{\tau=0}$ by $X^{s+\alpha,-\frac{1}{2}+\epsilon+}_{\tau=0}$ on the left hand side.

Proof of (25). As usual the regularity of $|\nabla|^{-1}$ is harmless in dimension $n \geq 3$ ([T], Cor. 8.2) and it can be replaced by $\langle\nabla\rangle^{-1}$. Taking care of the time derivative we reduce to

$$|\int\int u_1u_2u_3\,dxdt| \lesssim \|u_1\|_{X^{s+\alpha,\frac{1}{2}+\epsilon}_{\tau=0}} \|u_2\|_{X^{s+\alpha,-\frac{1}{2}+\epsilon}_{\tau=0}} \|u_3\|_{X^{1-(\alpha+s),\frac{1}{2}-2\epsilon+}_{\tau=0}} ,$$

which follows from Sobolev's multiplication rule, because under our assumption on s and the choice of α we obtain $2(s+\alpha) + 1 - (\alpha+s) > \frac{n}{2}$, as one easily calculates.

Proof of (26). a. If $\widehat{\phi}$ is supported in $||\tau| - |\xi|| \gtrsim |\xi|$, we obtain

$$\|\phi\|_{X^{s+\alpha,\frac{1}{2}+\epsilon}_{\tau=0}} \lesssim \|\phi\|_{X^{s,\frac{3}{4}+\epsilon}_{|\tau|=|\xi|}} ,$$

when we remark that $\alpha \leq \frac{1}{4}$ for $n \geq 3$. Thus (26) follows from (25).

b. It remains to show

$$|\int\int (uv_tw + uvw_t)\,dxdt| \lesssim \|u\|_{X^{1-\alpha-s,\frac{1}{2}-\epsilon}_{\tau=0}} \|w\|_{X^{s,\frac{3}{4}+\epsilon}_{|\tau|=|\xi|}} \|v\|_{X^{s+\alpha-\epsilon,\frac{1}{2}+\epsilon}_{\tau=0}}$$

whenever $\widehat{w}$ is supported in $||\tau|-|\xi|| \ll |\xi|$. This is equivalent to

$$\int_* m(\xi_1,\xi_2,\xi_3,\tau_1,\tau_2,\tau_3) \prod_{i=1}^3 \widehat{u}_i(\xi_i,\tau_i) d\xi d\tau \lesssim \prod_{i=1}^3 \|u_i\|_{L^2_{xt}}$$

where $d\xi = d\xi_1 d\xi_2 d\xi_3$, $d\tau = d\tau_1 d\tau_2 d\tau_n$ and * denotes integration over $\sum_{i=1}^3 \xi_i = \sum_{i=1}^3 \tau_i = 0$. The Fourier transforms are nonnegative without loss of generality. Here

$$m = \frac{(|\tau_2| + |\tau_3|)\chi_{||\tau_3|-|\xi_3|| \ll |\xi_3|}}{\langle \xi_1 \rangle^{1-\alpha-s} \langle \tau_1 \rangle^{\frac{1}{2}-\epsilon} \langle \xi_2 \rangle^{s+\alpha-\epsilon} \langle \tau_2 \rangle^{\frac{1}{2}+\epsilon} \langle \xi_3 \rangle^s \langle |\tau_3| - |\xi_3| \rangle^{\frac{3}{4}+\epsilon}} .$$

Since $\langle \tau_3 \rangle \sim \langle \xi_3 \rangle$ and $\tau_1 + \tau_2 + \tau_3 = 0$ we have

$$|\tau_2| + |\tau_3| \lesssim \langle \tau_1 \rangle^{\frac{1}{2}-\epsilon} \langle \tau_2 \rangle^{\frac{1}{2}+\epsilon} + \langle \tau_1 \rangle^{\frac{1}{2}-\epsilon} \langle \xi_3 \rangle^{\frac{1}{2}+\epsilon} + \langle \tau_2 \rangle^{\frac{1}{2}+\epsilon} \langle \xi_3 \rangle^{\frac{1}{2}-\epsilon}, \tag{33}$$

so that concerning the first term on the right hand side of (33) we have to show

$$|\int\int uvw dx dt| \lesssim \|u\|_{X^{1-\alpha-s,0}_{\tau=0}} \|v\|_{X^{s+\alpha-\epsilon,0}_{\tau=0}} \|w\|_{X^{s,\frac{3}{4}+\epsilon}_{|\tau|=|\xi|}} ,$$

which easily follows from Sobolev's multiplication rule, because $s > \frac{n}{2} - 1$.
Concerning the second term on the right hand side of (33) we use $\langle \xi_1 \rangle^{s-1+\alpha} \lesssim \langle \xi_2 \rangle^{s-1+\alpha} + \langle \xi_3 \rangle^{s-1+\alpha}$, so that we reduce to

$$|\int\int uvw dx dt| \lesssim \|u\|_{X^{0,0}_{\tau=0}} \|v\|_{X^{1-\epsilon,\frac{1}{2}+\epsilon}_{\tau=0}} \|w\|_{X^{s-\frac{1}{2}-\epsilon,\frac{3}{4}+\epsilon}_{|\tau|=|\xi|}} \tag{34}$$

and

$$|\int\int uvw dx dt| \lesssim \|u\|_{X^{0,0}_{\tau=0}} \|v\|_{X^{s+\alpha-\epsilon,\frac{1}{2}+\epsilon}_{\tau=0}} \|w\|_{X^{\frac{1}{2}-\alpha-\epsilon,\frac{3}{4}+\epsilon}_{|\tau|=|\xi|}} . \tag{35}$$

To obtain (34) in the case $n \ge 4$ we estimate as follows:

$$|\int\int uvw dx dt| \le \|u\|_{L^2_x L^2_t} \|v\|_{L_x^{\frac{2n}{n-2+2\epsilon}} L^\infty_t} \|w\|_{L_x^{\frac{n}{1-\epsilon}} L^2_t} .$$

We use (23) with $p = \frac{n}{1-\epsilon}$ and $k = n(\frac{n-1}{2(n+1)} - \frac{1}{p})$, so that one easily checks that

$$k + \frac{n-1}{2(n+1)} = n(\frac{n-1}{2(n+1)} - \frac{1-\epsilon}{n}) + \frac{n-1}{2(n+1)} < \frac{n}{2} - \frac{5}{4} < s - \frac{1}{2} - \epsilon .$$

Thus

$$\|w\|_{L_x^{\frac{n}{1-\epsilon}} L^2_t} \lesssim \|w\|_{X^{k+\frac{n-1}{2(n+1)},\frac{1}{2}+}_{|\tau|=|\xi|}} \le \|w\|_{X^{s-\frac{1}{2}-\epsilon,\frac{1}{2}+}_{|\tau|=|\xi|}}$$

and by Sobolev

$$\|v\|_{L_x^{\frac{2n}{n-2+2\epsilon}} L^\infty_t} \lesssim \|v\|_{X^{1-\epsilon,\frac{1}{2}+}_{\tau=0}} .$$

In the case $n=3$ we estimate by Sobolev and (23)

$$|\int\int uvw dx dt| \le \|u\|_{L^2_x L^2_t} \|v\|_{L^4_x L^\infty_t} \|w\|_{L^4_x L^2_t} \lesssim \|u\|_{X^{0,0}_{\tau=0}} \|v\|_{X^{1-\epsilon,\frac{1}{2}+}_{|\tau|=|\xi|}} \|w\|_{X^{\frac{1}{4},\frac{1}{2}+}_{|\tau|=|\xi|}}$$

In order to obtain (35) we estimate as follows:

$$|\int\int uvw dx dt| \le \|u\|_{L^2_x L^2_t} \|v\|_{L^{\tilde{q}}_x L^\infty_t} \|w\|_{L^p_x L^2_t}$$

with $\frac{1}{\tilde{q}} = \frac{2(\frac{1}{2}-\alpha-\epsilon)}{n-1}$ and $\frac{1}{p} = \frac{n-1-4(\frac{1}{2}-\alpha-\epsilon)}{2(n-1)}$. Then we use the embedding $H^{s+\alpha-\epsilon}_x \subset L^{\tilde{q}}_x$. This is true, because one easily checks $\frac{2(\frac{1}{2}-\alpha-\epsilon)}{n-1} \ge \frac{1}{2} - \frac{s+\alpha-\epsilon}{n}$, using $\alpha \le \frac{1}{4}$ and $s > \frac{n}{2} - \frac{3}{4}$. We next show that

$$\|w\|_{L^p_x L^2_t} \lesssim \|w\|_{X^{\frac{1}{2}-\alpha-\epsilon,\frac{1}{2}+}_{|\tau|=|\xi|}} .$$

This follows by interpolation between (23) (with $k=0$) and the trivial identity $\|w\|_{L^2_xL^2_t} = \|u\|_{X^{0,0}_{|\tau|=|\xi|}}$ with interpolation parameter θ given by $\theta\frac{n-1}{2(n+1)} = \frac{1}{2}-\alpha-\epsilon$. One checks that $\theta<1$ and $\frac{1}{p} = \frac{1-\theta}{2} + \theta\frac{n-1}{2(n+1)}$, so that (35) follows.

Concerning the last term on the right hand side of (33) we use $\langle\xi_1\rangle^{s-1+\alpha} \lesssim \langle\xi_2\rangle^{s-1+\alpha} + \langle\xi_3\rangle^{s-1+\alpha}$ so that we reduce to

$$|\int\int uvwdxdt| \lesssim \|u\|_{X^{0,\frac{1}{2}-\epsilon}_{\tau=0}} \|v\|_{X^{1-\epsilon,0}_{\tau=0}} \|w\|_{X^{s-\frac{1}{2}+\epsilon,\frac{3}{4}+\epsilon}_{|\tau|=|\xi|}} \tag{36}$$

and

$$|\int\int uvwdxdt| \lesssim \|u\|_{X^{0,\frac{1}{2}-\epsilon}_{\tau=0}} \|v\|_{X^{s+\alpha-\epsilon,0}_{\tau=0}} \|w\|_{X^{\frac{1}{2}-\alpha+\epsilon,\frac{3}{4}+\epsilon}_{|\tau|=|\xi|}} . \tag{37}$$

In order to obtain (36) in the case $n\ge 4$ we estimate by Hölder's inequality

$$|\int\int uvwdxdt| \le \|u\|_{L^2_xL^{\frac{1}{\epsilon}}_t} \|v\|_{L_x^{\frac{2n}{n-2+2\epsilon}}L^2_t} \|w\|_{L^{\frac{n}{1-\epsilon}}L_t^{\frac{2}{1-2\epsilon}}} .$$

By Sobolev we have

$$\|v\|_{L_x^{\frac{2n}{n-2+2\epsilon}}L^2_t} \lesssim \|v\|_{X^{1-\epsilon,0}_{\tau=0}} ,$$

and by (23) we obtain for $\frac{1}{p} = \frac{1}{n} - O(\epsilon)$:

$$\|w\|_{L^p_xL^2_t} \lesssim \|w\|_{X^{k+\frac{n-1}{2(n+1)},\frac{1}{2}+}_{|\tau|=|\xi|}} ,$$

where

$$\frac{k}{n} = \frac{n-1}{2(n+1)} - \frac{1}{n} + O(\epsilon) \Leftrightarrow k + \frac{n-1}{2(n+1)} = \frac{n}{2} - \frac{3}{2} + O(\epsilon) < s - \frac{3}{4} .$$

Interpolation with the standard Strichartz inequality (22) for $q=r=\frac{2(n+1)}{n-1}$:

$$\|w\|_{L_x^{\frac{2(n+1)}{n-1}}L_t^{\frac{2(n+1)}{n-1}}} = \|w\|_{L_t^{\frac{2(n+1)}{n-1}}L_x^{\frac{2(n+1)}{n-1}}} \lesssim \|w\|_{X^{\frac{1}{2},\frac{1}{2}+}_{|\tau|=|\xi|}}$$

and interpolation parameter $\theta=(n+1)\epsilon$ gives

$$\|w\|_{L_x^{\frac{n}{1-\epsilon}}L_t^{\frac{2}{1-2\epsilon}}} \lesssim \|w\|_{X^{s-\frac{3}{4},\frac{1}{2}+}_{|\tau|=|\xi|}} ,$$

which is more than we need.
In order to obtain (36) in the case $n=3$ we estimate as follows:

$$\begin{aligned} |\int\int uvwdxdt| &\lesssim \|u\|_{L^2_xL^{\frac{1}{\epsilon}}_t} \|v\|_{L^4_xL^2_t} \|w\|_{L^4L_t^{\frac{2}{1-2\epsilon}}} \\ &\lesssim \|u\|_{X^{0,\frac{1}{2}-\epsilon}_{\tau=0}} \|v\|_{X^{1-\epsilon,0}_{\tau=0}} \|w\|_{X^{\frac{1}{4}+\epsilon,\frac{1}{2}+\epsilon}_{|\tau|=|\xi|}} , \end{aligned}$$

which is sufficient under our assumption $s>\frac{3}{4}$.
In order to obtain (37) we estimate

$$|\int\int uvwdxdt| \le \|u\|_{L^2_xL^{\frac{1}{\epsilon}}_t} \|v\|_{L^p_xL^2_t} \|w\|_{L^q_xL_t^{\frac{2}{1-2\epsilon}}} ,$$

where $\frac{1}{p} = \frac{1}{2} - \frac{s+\alpha-\epsilon}{n}$ and $\frac{1}{q} = \frac{s+\alpha-\epsilon}{n}$, so that by Sobolev

$$\|v\|_{L^p_xL^2_t} \lesssim \|v\|_{X^{s+\alpha-\epsilon,0}_{\tau=0}} .$$

One easily checks that $\frac{1}{\tilde q} := \frac{1}{q} - O(\epsilon) > \frac{n-1}{2(n+1)}$ under our assumptions on s and α. By (23) we obtain

$$\|w\|_{L_x^{\frac{2(n+1)}{n-1}}L^2_t} \lesssim \|w\|_{X^{\frac{n-1}{2(n+1)},\frac{1}{2}+}_{|\tau|=|\xi|}} ,$$

which we interpolate with the trivial identity $\|w\|_{L^2_x L^2_t} = \|w\|_{X^{0,0}_{|\tau|=|\xi|}}$, where the interpolation parameter θ is chosen such that

$$\frac{1}{\tilde q} = \theta \frac{n-1}{2(n+1)} + (1-\theta)\frac{1}{2} \Leftrightarrow \theta = (n+1)(\frac{1}{2} - \frac{s+\alpha}{n}) + O(\epsilon) \, ,$$

we obtain

$$\|w\|_{L^{\tilde q}_x L^2_t} \lesssim \|w\|_{X^{k,\frac{1}{2}+}_{|\tau|=|\xi|}}$$

with $k = \theta \frac{n-1}{2(n+1)} = \frac{n-1}{2}(\frac{1}{2} - \frac{s+\alpha}{n}) + O(\epsilon)$. An easy calculation now shows that $k < \frac{1}{2} - \alpha$, so that another interpolation with Strichartz' inequality

$$\|w\|_{L_x^{\frac{2(n+1)}{n-1}} L_t^{\frac{2(n+1)}{n-1}}} \lesssim \|w\|_{X^{\frac{1}{2},\frac{1}{2}+}_{|\tau|=|\xi|}}$$

and interpolation parameter $\theta = (n+1)\epsilon$ gives

$$\|w\|_{L^q_x L_t^{\frac{2}{1-2\epsilon}}} \lesssim \|w\|_{X^{k+O(\epsilon),\frac{1}{2}+}_{|\tau|=|\xi|}} \lesssim \|w\|_{X^{\frac{1}{2}-\alpha+\epsilon,\frac{1}{2}+}_{|\tau|=|\xi|}} \, .$$

This completes the proof of (26).

Proof of (24). If $\widehat{\phi}$ is supported in $||\tau| - |\xi|| \gtrsim |\xi|$ we obtain

$$\|\phi\|_{X^{s+\alpha,\frac{1}{2}+\epsilon}_{\tau=0}} \lesssim \|\phi\|_{X^{s,\frac{3}{4}+\epsilon}_{|\tau|=|\xi|}} \, .$$

which implies that (24) follows from (26), if $\widehat{\phi}_1$ or $\widehat{\phi}_2$ have this support property. So we may assume that both functions are supported in $||\tau| - |\xi|| \ll |\xi|$. This means that it suffices to show

$$\int_* m(\xi_1,\xi_2,\xi_3,\tau_1,\tau_2,\tau_3) \prod_{i=1}^3 \widehat{u}_i(\xi_i,\tau_i) d\xi d\tau \lesssim \prod_{i=1}^3 \|u_i\|_{L^2_{xt}} \, ,$$

where

$$m = \frac{|\tau_3| \chi_{||\tau_2|-|\xi_2|| \ll |\xi_2|} \chi_{||\tau_3|-|\xi_3|| \ll |\xi_3|}}{\langle \xi_1 \rangle^{1-\alpha-s} \langle \tau_1 \rangle^{\frac{1}{2}-\epsilon-} \langle \xi_2 \rangle^s \langle |\tau_2| - |\xi_2| \rangle^{\frac{3}{4}+\epsilon} \langle \xi_3 \rangle^s \langle |\tau_3| - |\xi_3| \rangle^{\frac{3}{4}+\epsilon}} \, .$$

Since $\langle \tau_3 \rangle \sim \langle \xi_3 \rangle$, $\langle \tau_2 \rangle \sim \langle \xi_2 \rangle$ and $\tau_1 + \tau_2 + \tau_3 = 0$ we have

$$|\tau_3| \lesssim \langle \tau_1 \rangle^{\frac{1}{2}-\epsilon-} \langle \xi_3 \rangle^{\frac{1}{2}+\epsilon+} + \langle \xi_2 \rangle^{\frac{1}{2}-\epsilon-} \langle \xi_3 \rangle^{\frac{1}{2}+\epsilon+}, \tag{38}$$

Concerning the first term on the right hand side we have to show

$$\Big| \int \int uvw dx dt \Big| \lesssim \|u\|_{X^{1-\alpha-s,0}_{\tau=0}} \|v\|_{X^{s,\frac{3}{4}+\epsilon}_{|\tau|=|\xi|}} \|w\|_{X^{s-\frac{1}{2}-\epsilon-,\frac{3}{4}+\epsilon}_{|\tau|=|\xi|}} \, .$$

We use [FK] ,Thm. 1.1 , which shows

$$\|vw\|_{L^2_t H^{s-\frac{3}{4}}_x} \lesssim \|v\|_{X^{s,\frac{1}{2}+}_{|\tau|=|\xi|}} \|w\|_{X^{s-\frac{1}{2}-\epsilon-,\frac{1}{2}+}_{|\tau|=|\xi|}}$$

under the assumption $s > \frac{n}{2} - \frac{3}{4}$. This is enough, because $\alpha \le \frac{1}{4}$.

Concerning the second term on the right hand side we use $\langle \xi_1 \rangle^{s-1+\alpha} \lesssim \langle \xi_2 \rangle^{s-1+\alpha} + \langle \xi_3 \rangle^{s-1+\alpha}$, so that we reduce to

$$\Big| \int \int uvw dx dt \Big| \lesssim \|u\|_{X^{0,\frac{1}{2}-\epsilon-}_{\tau=0}} \|v\|_{X^{\frac{1}{2}-\alpha+\epsilon+,\frac{3}{4}+\epsilon}_{|\tau|=|\xi|}} \|w\|_{X^{s-\frac{1}{2}-\epsilon-,\frac{3}{4}+\epsilon}_{|\tau|=|\xi|}}$$

and

$$\Big| \int \int uvw dx dt \Big| \lesssim \|u\|_{X^{0,\frac{1}{2}-\epsilon}_{\tau=0}} \|v\|_{X^{s-\frac{1}{2}+\epsilon+,\frac{3}{4}+\epsilon}_{|\tau|=|\xi|}} \|w\|_{X^{\frac{1}{2}-\alpha-\epsilon-,\frac{3}{4}+\epsilon}_{|\tau|=|\xi|}} \, .$$

We even show the slightly stronger estimate

$$\Big| \int \int uvw dx dt \Big| \lesssim \|u\|_{X^{0,\frac{1}{2}-\epsilon-}_{\tau=0}} \|v\|_{X^{\frac{1}{2}-\alpha-\epsilon-,\frac{3}{4}+\epsilon}_{|\tau|=|\xi|}} \|w\|_{X^{s-\frac{1}{2}-\epsilon-,\frac{3}{4}+\epsilon}_{|\tau|=|\xi|}} \, ,$$

which implies both. We start with the estimate

$$|\int\int uvwdxdt| \lesssim \|u\|_{L^2_x L_t^{\frac{1}{\epsilon}-}} \|v\|_{L^p_x L_t^{\frac{2}{1-2\epsilon}+}} \|w\|_{L^q_x L^2_t} \, ,$$

where $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. Interpolating (23)

$$\|v\|_{L_x^{\frac{2(n+1)}{n-1}} L^2_t} \lesssim \|v\|_{X^{\frac{n-1}{2(n+1)},\frac{1}{2}+}_{|\tau|=|\xi|}}$$

with the trivial identity $\|v\|_{L^2_x L^2_t} = \|v\|_{X^{0,0}_{|\tau|=|\xi|}}$ with interpolation parameter θ given by $\theta \frac{n-1}{2(n+1)} = \frac{1}{2} - \alpha - 2\epsilon$ (where we remark that $\theta < 1$) this gives

$$\|v\|_{L^{\tilde{p}}_x L^2_t} \lesssim \|v\|_{X^{\frac{1}{2}-\alpha-O(\epsilon),\frac{1}{2}+}_{|\tau|=|\xi|}} \, ,$$

where

$$\frac{1}{\tilde{p}} = \frac{n-1}{2(n+1)}\theta + (1-\theta)\frac{1}{2} = \frac{n-3}{2(n-1)} + \frac{2}{n-1}\alpha + O(\epsilon) \, .$$

Interpolating this estimate with Strichartz' estimate just slightly changing the parameters we obtain

$$\|v\|_{L^p_x L_x^{\frac{2}{1-2\epsilon}+}} \lesssim \|v\|_{X^{\frac{1}{2}-\alpha-\epsilon-,\frac{1}{2}+}_{|\tau|=|\xi|}} \, ,$$

where $\frac{1}{p} = \frac{1}{\tilde{p}} + O(\epsilon)$. Thus $\frac{1}{q} = \frac{1}{2} - \frac{1}{p} = \frac{1}{n-1} - \frac{2}{n-1}\alpha - O(\epsilon)$.
Next we apply (23) to obtain

$$\|w\|_{L^q_x L^2_t} \lesssim \|w\|_{X^{k+\frac{n-1}{2(n+1)},\frac{1}{2}+}_{|\tau|=|\xi|}}$$

with

$$\frac{1}{q} = \frac{n-1}{2(n+1)} - \frac{k}{n} \Leftrightarrow k = n(\frac{n-1}{2(n+1)} - \frac{1}{n-1} + \frac{2}{n-1}\alpha) + O(\epsilon) \, .$$

In order to conclude the desired estimate

$$\|w\|_{L^q_x L^2_t} \lesssim \|w\|_{X^{s-\frac{1}{2}-\epsilon-,\frac{1}{2}+}_{|\tau|=|\xi|}}$$

we need

$$s \geq k + \frac{n}{n+1} + O(\epsilon) = \frac{n}{2} - \frac{n}{n-1} + \frac{2n}{n-1}\alpha + O(\epsilon) \, . \tag{39}$$

This means that in order to obtain a minimal lower bound for s one should also minimize α. On the other hand in the proof of (30) below we have to maximize α. Comparing condition (39) with (42) below we optimize α by choosing

$$\frac{n}{2} - \frac{n}{n-1} + \frac{2n}{n-1}\alpha = \frac{n}{2} - \frac{1}{4} - 2\alpha \Leftrightarrow \alpha = \frac{3n+1}{8(2n-1)} \, , \tag{40}$$

which leads to our choice of α. Thus the condition on s reduces to

$$s \geq \frac{n}{2} - \frac{1}{4} - 2\alpha + O(\epsilon) = \frac{n}{2} - \frac{5}{8} - \frac{5}{8(2n-1)} + O(\epsilon) \, .$$

This is exactly our assumption on s.
Proof of (27): Sobolev's multiplication law shows the estimate

$$\||\nabla|^{-1}(A_1 \partial_t A_2)\|_{C^0(H^{s-1})} \lesssim \|A_1\|_{C^0(H^s)} \|\partial_t A_2\|_{C^0(H^{s-1})}$$

for $s > \frac{n}{2} - 1$. Use now

$$A = A^{cf} + \sum_{\pm} A^{df}_{\pm} \quad , \quad \partial_t A = \partial_t A^{cf} + i\langle\nabla\rangle(A^{df}_+ - A^{df}_-)$$

from which the estimate (27) easily follows.

Proof of (29): This a generalization of the proof given by Tao ([T1]) in dimension $n=3$. We have to show

$$\int_* m(\xi,\tau)\prod_{i=1}^3 \widehat{u}_i(\xi_i,\tau_i)d\xi d\tau \lesssim \prod_{i=1}^3 \|u_i\|_{L^2_{xt}}\,,$$

where $\xi = (\xi_1,\xi_2,\xi_3)$, $\tau = (\tau_1,\tau_2,\tau_3)$, $*$ denotes integration over $\sum_{i=1}^3 \xi_i = \sum_{i=1}^3 \tau_i = 0$, and

$$m = \frac{(|\xi_2|+|\xi_3|)\langle\xi_1\rangle^{s-1}\langle|\tau_1|-|\xi_1|\rangle)^{-\frac14+2\epsilon}}{\langle\xi_2\rangle^s\langle|\tau_2|-|\xi_2|\rangle^{\frac34+\epsilon}\langle\xi_3\rangle^{s+\alpha}\langle\tau_3\rangle^{\frac12+\epsilon}}\,.$$

Case 1: $|\xi_2| \le |\xi_1|$ ($\Rightarrow |\xi_2|+|\xi_3| \lesssim |\xi_1|$).

By two applications of the averaging principle ([T], Prop. 5.1) we may replace m by

$$m' = \frac{\langle\xi_1\rangle^s\chi_{||\tau_2|-|\xi_2||\sim 1}\chi_{|\tau_3|\sim 1}}{\langle\xi_2\rangle^s\langle\xi_3\rangle^{s+\alpha}}\,.$$

Let now τ_2 be restricted to the region $\tau_2 = T+O(1)$ for some integer T. Then τ_1 is restricted to $\tau_1 = -T+O(1)$, because $\tau_1+\tau_2+\tau_3 = 0$, and ξ_2 is restricted to $|\xi_2| = |T|+O(1)$. The τ_1-regions are essentially disjoint for $T\in\mathbb{Z}$ and similarly the τ_2-regions. Thus by Schur's test ([T], Lemma 3.11) we only have to show

$$\sup_{T\in\mathbb{Z}}\int_* \frac{\langle\xi_1\rangle^s\chi_{\tau_1=-T+O(1)}\chi_{\tau_2=T+O(1)}\chi_{|\tau_3|\sim 1}\chi_{|\xi_2|=|T|+O(1)}}{\langle\xi_2\rangle^s\langle\xi_3\rangle^{s+\alpha}}\prod_{i=1}\widehat{u}_i(\xi_i,\tau_i)d\xi d\tau \lesssim \prod_{i=1}^3\|u_i\|_{L^2_{xt}}\,.$$

The τ-behaviour of the integral is now trivial, thus we reduce to

$$\sup_{T\in\mathbb{N}}\int_{\sum_{i=1}^3\xi_i=0}\frac{\langle\xi_1\rangle^s\chi_{|\xi_2|=|T|+O(1)}}{\langle T\rangle^s\langle\xi_3\rangle^{s+\alpha}}\widehat{f}_1(\xi_1)\widehat{f}_2(\xi_2)\widehat{f}_3(\xi_3)d\xi \lesssim \prod_{i=1}^3\|f_i\|_{L^2_x}\,. \tag{41}$$

Assuming now $|\xi_3|\le|\xi_1|$ (the other case being simpler) it only remains to consider the following two cases:

Case 1.1: $|\xi_1|\sim|\xi_3|\gtrsim T$. We obtain in this case

$$\begin{aligned} L.H.S.\,of\,(41) &\lesssim \sup_{T\in\mathbb{N}}\frac{1}{T^{s+\alpha}}\|f_1\|_{L^2}\|f_3\|_{L^2}\|\mathcal{F}^{-1}(\chi_{|\xi|=T+O(1)}\widehat{f}_2)\|_{L^\infty(\mathbb{R}^n)} \\ &\lesssim \sup_{T\in\mathbb{N}}\frac{1}{T^{s+\alpha}}\|f_1\|_{L^2}\|f_3\|_{L^2}\|\chi_{|\xi|=T+O(1)}\widehat{f}_2\|_{L^1(\mathbb{R}^n)} \\ &\lesssim \sup_{T\in\mathbb{N}}\frac{T^{\frac{n-1}{2}}}{T^{s+\alpha}}\prod_{i=1}^3\|f_i\|_{L^2} \lesssim \prod_{i=1}^3\|f_i\|_{L^2}\,, \end{aligned}$$

because one easily calculates that $2(s+\alpha) > n-1$ under our choice of s and α.

Case 1.2: $|\xi_1|\sim T\gtrsim|\xi_3|$. An elementary calculation shows that

$$L.H.S.\,of\,(41) \lesssim \sup_{T\in\mathbb{N}}\|\chi_{|\xi|=T+O(1)} * \langle\xi\rangle^{-2(s+\alpha)}\|^{\frac12}_{L^\infty(\mathbb{R}^{n-1})}\prod_{i=1}^3\|f_i\|_{L^2_x} \lesssim \prod_{i=1}^3\|f_i\|_{L^2_x}\,,$$

using as in case 1.1 that $2(s+\alpha) > n-1$, so that the desired estimate follows.

Case 2. $|\xi_1|\le|\xi_2|$ ($\Rightarrow |\xi_2|+|\xi_3|\lesssim|\xi_2|$).

Exactly as in case 1 we reduce to

$$\sup_{T\in\mathbb{N}}\int_{\sum_{i=1}^3\xi_i=0}\frac{\langle\xi_1\rangle^{s-1}\chi_{|\xi_2|=|T|+O(1)}}{\langle T\rangle^{s-1}\langle\xi_3\rangle^{s+\alpha}}\widehat{f}_1(\xi_1)\widehat{f}_2(\xi_2)\widehat{f}_2(\xi_3)d\xi \lesssim \prod_{i=1}^3\|f_i\|_{L^2_x}\,.$$

This can be treated as in case 1.
Proof of (30): By Sobolev's multiplication law we obtain

$$|\int\int fghdxdt| \lesssim \|f\|_{X^{s+\alpha,\frac{1}{2}+\epsilon}_{\tau=0}} \|g\|_{X^{s+\alpha-1,\frac{1}{2}+\epsilon}_{\tau=0}} \|h\|_{X^{-s+\frac{5}{4}-2\epsilon,-\frac{1}{2}}_{\tau=0}} ,$$

where we need that

$$s + 2\alpha + \frac{1}{4} - 2\epsilon > \frac{n}{2} , \tag{42}$$

which holds under our assumptions on s and α. Using the elementary estimate

$$\frac{\langle\xi\rangle^{\frac{1}{4}-2\epsilon}}{\langle\tau\rangle^{\frac{1}{4}-2\epsilon}} \lesssim \langle|\tau|-|\xi|\rangle^{\frac{1}{4}-2\epsilon}$$

we obtain

$$\|h\|_{X^{-s+\frac{5}{4}-2\epsilon,-\frac{1}{2}}_{\tau=0}} \lesssim \|h\|_{[X^{1-s,\frac{1}{4}-2\epsilon}_{|\tau|=|\xi|}}$$

which implies (30).
Proof of (31): We use the following consequences of Sobolev's embedding and Strichartz' inequality:

$$\|A\|_{L^\infty_t H^{s+\alpha}_x} \lesssim \|A\|_{X^{s+\alpha,\frac{1}{2}+}_{\tau=0}} , \tag{43}$$

$$\|A\|_{L^{4-}_t H^{1-s}_x} \lesssim \|A\|_{X^{1-s,\frac{1}{4}-}_{|\tau|=|\xi|}} \tag{44}$$

$$\|A\|_{L^4_t H^{s-\frac{n+1}{4(n-1)},\frac{2(n-1)}{n-2}}_x} \lesssim \|A\|_{X^{s,\frac{1}{2}+}_{|\tau|=|\xi|}} , \tag{45}$$

where we applied (22) with $q=4$, $r=\frac{2(n-1)}{n-2}$, $\mu=\frac{n+1}{4(n-1)}$ and also

$$\|A\|_{L^{4+}_t H^{s-\frac{n+1}{4(n-1)}+,\frac{2(n-1)}{n-2}-}_x} \lesssim \|A\|_{X^{s,\frac{1}{2}+}_{|\tau|=|\xi|}} . \tag{46}$$

Assume now $s\ge 1$. Taking the dual of (44) we obtain

$$\|A_1A_2A_3\|_{X^{s-1,-\frac{1}{4}+}_{|\tau|=|\xi|}} \lesssim \|A_1A_2A_3\|_{L^{\frac{4}{3}+}_t H^{s-1}_x} .$$

This can be estimated by

$$\|A_1\|_{L^{4+}_t H^{s-1,p-}_x} \|A_2\|_{L^4_t L^{\frac{2n}{1+\alpha}+}_x} \|A_3\|_{L^4_t L^{\frac{2n}{1+\alpha}+}_x}$$

where $\frac{1}{p}=\frac{1}{2}-\frac{1+\alpha}{n}$, and similar terms with reversed roles of A_j . Now by Sobolev we have $H^{s+\alpha,2}_x \subset H^{s-1,p-}_x$, so that

$$\|A_1\|_{L^{4+}_t H^{s-1,p-}_x} \lesssim \|A_1\|_{X^{s+\alpha,\frac{1}{2}+}_{\tau=0}} .$$

Next we obtain $H_x^{s-\frac{n+1}{4(n-1)}+,\frac{2(n-1)}{n-2}-} \subset H^{s-1,p-}_x$, because

$$\frac{1}{p} > \frac{n-2}{2(n-1)} - \frac{1}{n}(1-\frac{n+1}{4(n-1)}) \Leftrightarrow \frac{3n+1}{4n(n-1)} > \frac{\alpha}{n} ,$$

which holds, because $\frac{1}{4} \ge \alpha = \frac{3n+1}{8(2n-1)} \ge \frac{3}{16}$. This implies by (46)

$$\|A_1\|_{L^{4+}_t H^{s-1,p-}_x} \lesssim \|A_1\|_{X^{s,\frac{1}{2}+}_{|\tau|=|\xi|}} .$$

Next we have $H^{s+\alpha,2}_x \subset L_x^{\frac{2n}{1+\alpha}+}$, because the inequality

$$\frac{1+\alpha}{2n} > \frac{1}{2} - \frac{s+\alpha}{n}$$

holds by $s>\frac{n}{2}-\frac{3}{4}$ and $\alpha \ge \frac{3}{16}$. This implies

$$\|A_j\|_{L^4_t L^{\frac{2n}{1+\alpha}+}_x} \lesssim \|A_j\|_{L^4_t H^{s+\alpha,2}_x} \lesssim \|A_j\|_{X^{s+\alpha,\frac{1}{2}+}_{\tau=0}} .$$

Finally by Sobolev $H_x^{s-\frac{n+1}{4(n-1)},\frac{2(n-1)}{n-2}} \subset L_x^{\frac{2n}{1+\alpha}+}$, because one easily calculates that $\frac{1+\alpha}{2n} > \frac{n-2}{2(n-1)} - \frac{1}{n}(s - \frac{n+1}{4(n-1)})$ using $s \ge \frac{n}{2} - \frac{3}{4}$ and $\alpha \ge \frac{3}{16}$. Thus by (45)

$$\|A_j\|_{L_t^4 L_x^{\frac{2n}{1+\alpha}+}} \lesssim \|A_j\|_{L_t^4 H_x^{s-\frac{n+1}{4(n-1)},\frac{2(n-1)}{n-2}}} \lesssim \|A_j\|_{X^{s,\frac{1}{2}+}_{|\tau|=|\xi|}}.$$

This completes the proof of (31) for $s \ge 1$. It remains to consider the case $1 > s > \frac{3}{4}$ in dimension $n = 3$ and $\alpha = \frac{1}{4}$. This case is much easier. We only use

$$\|A\|_{L_t^{4-} L_x^2} \lesssim \|A\|_{X^{0,\frac{1}{4}-}_{|\tau|=|\xi|}} \lesssim \|A\|_{X^{1-s,\frac{1}{4}-}_{|\tau|=|\xi|}},$$

so that by duality

$$\|A_1 A_2 A_3\|_{X^{s-1,-\frac{1}{4}+}_{|\tau|=|\xi|}} \lesssim \|A_1 A_2 A_3\|_{L_t^{\frac{4}{3}+} L_x^2} \lesssim \prod_{I=1}^{3} \|A_i\|_{L_t^{4+} L_x^6}.$$

Now by Sobolev for $s > \frac{3}{4}$ we obtain

$$\|A_i\|_{L_t^{4+} L_x^6} \lesssim \|A_i\|_{L_t^{4+} H_x^1} \lesssim \|A_i\|_{X^{s+\frac{1}{4},\frac{1}{2}+}_{\tau=0}},$$

and using Sobolev's embedding and Strichartz' inequality (22) gives

$$\|A_i\|_{L_t^{4+} L_x^6} \lesssim \|A_i\|_{L_t^{4+} H_x^{\frac{1}{4}+,4-}} \lesssim \|A_i\|_{X^{\frac{3}{4}+,\frac{1}{2}+}_{|\tau|=|\xi|}} \lesssim \|A_i\|_{X^{s,\frac{1}{2}+}_{|\tau|=|\xi|}}.$$

Proof of (32): The case $N = 3$ reduces to (31). Next we consider the case $N = 4$ in dimension $n = 3$. We may assume $s \le 1$, because the general case can be reduced to this case easily. This follows from Prop. 3.3 as follows:

$$\||\phi|^3\phi\|_{X^{s-1,-\frac{1}{4}+}_{|\tau|=|\xi|}} \lesssim \||\phi|^3\phi\|_{L_t^{\frac{4}{3}+} H_x^{s-1}} \lesssim \||\phi|^3\phi\|_{L_t^{\frac{4}{3}+} L_x^p} \lesssim \|\phi\|^4_{L_t^{\frac{16}{3}+} L_x^{4p}},$$

where $\frac{1}{p} = \frac{1}{2} - \frac{s-1}{3}$. We now use Strichartz estimate (22) with $q = \frac{16}{3}+$, $r = \frac{16}{5}-$, $\mu = \frac{3}{8}+$ to conclude

$$\||\phi|^3\phi\|_{X^{s-1,-\frac{1}{4}+}_{|\tau|=|\xi|}} \lesssim \|\phi\|^4_{L_t^{\frac{16}{3}+} H_x^{l,\frac{16}{5}-}} \lesssim \|\phi\|^4_{X^{l+\mu,\frac{1}{2}+}_{|\tau|=|\xi|}} \lesssim \|\phi\|^4_{X^{s,\frac{1}{2}+}_{|\tau|=|\xi|}},$$

provided $H_x^{l,\frac{16}{5}-} \subset L_x^{4p}$, which is fulfilled, if $l = \frac{5+4s}{16}+$, so that $l + \mu \le s$, if $\frac{5+4s}{16} + \frac{3}{8} < s \Leftrightarrow s > \frac{11}{12}$, which is equivalent to our assumption $N < 1 + \frac{7}{4(\frac{n}{2}-s)}$. The case $N = 2$ for $n = 3$ is much easier handled by the standard Strichartz inequality:

$$\||\phi|\phi\|_{H^{s-1,-\frac{1}{4}+}_{|\tau|=|\xi|}} \lesssim \|\phi\|^2_{L_t^4 L_x^4} \lesssim \|\phi\|_{X^{\frac{1}{2},\frac{1}{2}+}_{|\tau|=|\xi|}}.$$

In all the other cases under our assumptions we have $s \ge 1$. We have

$$\||\phi|^{N-1}\phi\|_{X^{s-1,-\frac{1}{4}+}_{|\tau|=|\xi|}} \lesssim \||\phi|^{N-1}\phi\|_{L_t^{\frac{4}{3}+} H_x^{s-1}} \lesssim \|\phi\|^{N-1}_{L_t^{\frac{4}{3}N+} L_x^{\tilde{q}}} \|\phi\|_{L_t^{\frac{4}{3}N+} H_x^{s-1,p}}.$$

Here $\frac{1}{p} + \frac{N-1}{\tilde{q}} = \frac{1}{2}$. We obtain $H^{s-1,p} \subset L^{\tilde{q}}$, if $\frac{1}{\tilde{q}} = \frac{1}{p} - \frac{s-1}{n}$, so that

$$\frac{1}{p} = \frac{1}{N}\Big(\frac{1}{2} + \frac{(N-1)(s-1)}{n}\Big),$$

thus

$$\||\phi|^{N-1}\phi\|_{X^{s-1,-\frac{1}{4}+}_{|\tau|=|\xi|}} \lesssim \|\phi\|^N_{L_t^{\frac{4}{3}N+} H_x^{s-1,p}}.$$

The case $N = 2$ is again easy. In this case we have $\frac{1}{p} = \frac{s-1}{2n} + \frac{1}{4}$, which implies by Sobolev $H^{s,2} \subset H^{s-1,p}$ under the condition $\frac{1}{p} \ge \frac{1}{2} - \frac{1}{n}$, which is easily seen to be equivalent to $s \ge \frac{n}{2} - 1$, which certainly holds, so that we obtain the desired bound $\|\phi\|^2_{X^{s,\frac{1}{2}+}_{|\tau|=|\xi|}}$.

It remains to consider $N \ge 4$. We use Strichartz' estimate (22) with $q = \frac{4}{3}N+$, $\frac{1}{r} = \frac{1}{2} - \frac{3}{2N(n-1)}+$, $\mu = n(\frac{1}{2} - \frac{1}{r}) - \frac{1}{q} = \frac{3(n+1)}{4N(n-1)}+$ to conclude

$$\| |\phi|^{N-1}\phi\|_{X^{s-1,-\frac{1}{4}+}_{|\tau|=|\xi|}} \lesssim \|\phi\|^N_{L_t^{\frac{4}{3}N+} H_x^{l,r}} \lesssim \|\phi\|^N_{X^{l+\mu,\frac{1}{2}+}_{|\tau|=|\xi|}} \lesssim \|\phi\|^N_{X^{s,\frac{1}{2}+}_{|\tau|=|\xi|}} , \qquad (47)$$

if we $H^{l,r} \subset H^{s-1,p}$ and $l + \mu \le s$. By Sobolev we need

$$\frac{1}{r} \ge \frac{1}{p} \ge \frac{1}{r} - \frac{l-s+1}{n} .$$

We calculate

$$\begin{aligned} \frac{1}{r} \ge \frac{1}{p} &\Leftrightarrow \frac{1}{2} - \frac{3}{2N(n-1)} > \frac{1}{N}\Big(\frac{1}{2} + \frac{(N-1)(s-1)}{n}\Big) \\ &\Leftrightarrow s < \frac{n}{2} + 1 - \frac{3n}{(N-1)2(n-1)} . \end{aligned} \qquad (48)$$

In this case we can choose $l = \frac{n}{r} - \frac{n}{p} + s - 1$, so that one easily calculates

$$\begin{aligned} & l + \mu \le s \\ & \Leftrightarrow n\Big(\frac{1}{2} - \frac{3}{2N(n-1)}\Big) - \frac{n}{N}\Big(\frac{1}{2} + \frac{(N-1)(s-1)}{n}\Big) + s - 1 + \frac{3(n+1)}{4N(n-1)} < s \\ & \Leftrightarrow s > \frac{n}{2} - \frac{7}{4(N-1)} \Leftrightarrow \Big(N < 1 + \frac{7}{4(\frac{n}{2}-s)} \text{ if } s < \frac{n}{2} \text{ and } N < \infty \text{ if } s \ge \frac{n}{2}\Big) . \end{aligned}$$

This is exactly our assumption on s and N. This lower bound on s and also the lower bound on s in Prop 3.2 is compatible with the upper bound (48) in our case $N \ge 4$ and $n \ge 3$, as an easy calculation shows. As always the desired estimate (47) for greater s can be reduced to this case so that (48) is redundant. Thus (47) is proven. This completes the proof of (32) and also the proof of Prop. 3.2 and Prop. 3.1. $\square$

4. Removal of the assumption $A^{cf}(0) = 0$

Applying an idea of Keel and Tao [T1] we use the gauge invariance of the Yang-Mills-Higgs system to show that the condition $A^{cf}(0) = 0$, which had to be assumed in Prop. 3.2, can be removed. A completely analogous result holds for the Yang-Mills equation and Prop. 3.1.

Lemma 4.1. *Let $n \ge 3$, $s > \frac{n}{2} - \frac{3}{4}$ and $0 < \epsilon \ll 1$. Assume $(A,\phi) \in (C^0([0,1],H^s) \cap C^1([0,1],H^{s-1}) \times (C^0([0,1],H^s) \cap C^1([0,1],H^{s-1}))$, $A_0 = 0$ and*

$$\|A^{df}(0)\|_{H^s} + \|(\partial_t A)^{df}(0)\|_{H^{s-1}} + \|A^{cf}(0)\|_{H^s} + \|\phi(0)\|_{H^s} + \|(\partial_t\phi)(0)\|_{H^{s-1}} \le \epsilon . \qquad (49)$$

Then there exists a gauge transformation T preserving the temporal gauge such that $(TA)^{cf}(0) = 0$ and

$$\|(TA)^{df}(0)\|_{H^s} + \|(\partial_t TA)^{df}(0)\|_{H^{s-1}} + \|(T\phi)(0)\|_{H^s} + \|(\partial_t T\phi)(0)\|_{H^{s-1}} \lesssim \epsilon . \qquad (50)$$

T preserves also the regularity, i.e. $TA \in C^0([0,1],H^s) \cap C^1([0,1],H^{s-1})$, $T\phi \in C^0([0,1],H^s) \cap C^1([0,1],H^{s-1})$. If $A \in X_+^{s,\frac{3}{4}+}[0,1] + X_-^{s,\frac{3}{4}+}[0,1] + X_{\tau=0}^{s+\alpha,\frac{1}{2}+}[0,1]$, where $\alpha = \frac{3n+1}{8(2n-1)}$, $\partial_t A^{cf} \in C^0([0,1],H^{s-1})$ and $\phi \in X_+^{s,\frac{3}{4}+}[0,1] + X_-^{s,\frac{3}{4}+}[0,1]$, then TA , $T\phi$ belong to the same spaces. Its inverse T^{-1} has the same properties.

In the proof we frequently use

Lemma 4.2. *Let $n \ge 3$, $s > \frac{n}{2} - 1$ and define $\|f\|_X := \|\nabla f\|_{H^s}$. The following estimates hold:*

$$\begin{aligned}\|fg\|_X &\le c_1 \|f\|_X \|g\|_X \\ \|fg\|_{H^s} &\le c_1 \|f\|_X \|g\|_{H^s} \\ \|fg\|_{H^{s-1}} &\le c_1 \|f\|_X \|g\|_{H^{s-1}} \,.\end{aligned}$$

Proof. This follows essentially by Sobolev's multiplication law, where we remark that the singularity of $|\nabla|^{-1}$ is harmless in dimension $n \ge 3$. □

Proof of Lemma 4.1. This is achieved by an iteration argument. Assume that one has besides (49):

$$\|A^{cf}(0)\|_{H^s} \le \delta \tag{51}$$

for some $0 < \delta \le \epsilon$. In the first step we set $\delta = \epsilon$, so that the condition is fulfilled, in the next steps $\delta = \epsilon^{\frac{3}{2}}$, $\delta = \epsilon^2$ etc. We use the Hodge decomposition of A:

$$A = A^{cf} + A^{df} = (-\Delta)^{-1} \nabla \, div \, A + A^{df} \,.$$

We define $V_1 := -(-\Delta)^{-1} div \, A(0)$, so that $\nabla V_1 = A^{cf}(0)$. Thus

$$\|V_1\|_X := \|\nabla V_1\|_{H^s} = \|A^{cf}(0)\|_{H^s} \le \delta \,.$$

We define $U_1 := \exp(V_1)$ and consider the gauge transformation T_1 with

$$\begin{aligned}A_0 &\longmapsto U_1 A_0 U_1^{-1} - (\partial_t U_1) U_1^{-1} \\ A &\longmapsto U_1 A U_1^{-1} - (\nabla U_1) U_1^{-1} \\ \phi &\longmapsto U_1 \phi U_1^{-1} \,.\end{aligned}$$

Then T_1 preserves the temporal gauge, because U_1 is independent of t, a property, which is true for all the gauge transformations in the sequel as well. Moreover

$$\begin{aligned}(T_1 A)(0) &= \exp V_1 A(0) \exp(-V_1) - \nabla(\exp V_1) \exp(-V_1) \\ &= A^{df}(0) + (\exp V_1 A^{df}(0) \exp(-V_1) - A^{df}(0)) \\ &\quad + (\exp V_1 A^{cf}(0) - \nabla(\exp V_1)) \exp(-V_1)\end{aligned} \tag{52}$$

and thus

$$\begin{aligned}(T_1 A)^{cf}(0) &= -(-\Delta)^{-1} \nabla div (\exp V_1 A^{df}(0) \exp(-V_1) - A^{df}(0)) \\ &\quad - (-\Delta)^{-1} \nabla div ((\exp V_1 A^{cf}(0) - \nabla(\exp V_1)) \exp(-V_1)) \,.\end{aligned} \tag{53}$$

Using a Taylor expansion and Lemma 4.1 we obtain

$$\begin{aligned}&\| \exp V_1 A^{df}(0) \exp(-V_1) - A^{df}(0)\|_{H^s} \\ &\lesssim \|(\exp V_1 - I) A^{df}(0)(\exp(-V_1) - I)\|_{H^s} + \|A^{df}(0)(\exp(-V_1) - I)\|_{H^s} \\ &\quad + \|(\exp V_1 - I) A^{df}(0)\|_{H^s} \\ &\lesssim (\|\exp V_1 - I\|_X + 1) \|A^{df}(0)\|_{H^s} \| \exp(-V_1) - I\|_X \\ &\quad + \|\exp V_1 - I\|_X \|A^{df}(0)\|_{H^s} \\ &\lesssim (1+\delta)\epsilon\delta \\ &\le \frac{c_0}{2} \epsilon \delta \,.\end{aligned}$$

We used the estimate

$$\begin{aligned}\|\exp V_1 - I\|_X &\le \sum_{k=1}^{\infty} \frac{\|V_1^k\|_X}{k!} \le \sum_{k=1}^{\infty} \frac{(c_1 \|V_1\|_X)^k}{c_1 k!} = c_1^{-1}(\exp(c_1 \|V_1\|_X) - 1) \\ &\le c_1^{-1}(\exp(c_1 \delta) - 1) \lesssim \delta \,.\end{aligned}$$

Furthermore we obtain

$$\begin{aligned}
&\| \exp V_1 A^{cf}(0) - \nabla(\exp V_1))\|_{H^s} = \| \sum_{k=0}^{\infty} \frac{V_1^k}{k!} \nabla V_1 - \sum_{k=1}^{\infty} \frac{\nabla(V_1^k)}{k!} \|_{H^s} \\
&= \| \sum_{k=1}^{\infty} \frac{V_1^k}{k!} \nabla V_1 - \sum_{k=2}^{\infty} \frac{\nabla(V_1^k)}{k!} \|_{H^s} \lesssim \sum_{k=1}^{\infty} \frac{\|V_1^k\|_X}{k!} \|\nabla V_1\|_{H^s} + \sum_{k=2}^{\infty} \frac{\|\nabla(V_1^k)\|_{H^s}}{k!} \\
&\lesssim \sum_{k=1}^{\infty} \frac{c_1^k \|V_1\|_X^k}{k!} \|\nabla V_1\|_{H^s} + \sum_{k=2}^{\infty} \frac{c_1^k \|V_1\|_X^k}{k!} \\
&\lesssim (\exp(c_1 \|V_1\|_X) - 1) \|\nabla V_1\|_{H^s} + (\exp(c_1\|V_1\|_X) - 1 - c_1 \|V_1\|_X) \\
&\le \frac{c_0}{2} \delta^2 \, .
\end{aligned}$$

These estimates imply bv (53) in the case $\delta = \epsilon \ll 1$:

$$\|(T_1 A)^{cf}(0)\|_{H^s} \lesssim c_0 \epsilon \delta = c_0 \epsilon^2 \le \frac{1}{2} \epsilon^{\frac{3}{2}} \, . \tag{54}$$

Moreover by (52)

$$\|(T_1 A)(0)\|_{H^s} \le \|A^{df}(0)\|_{H^s} + c_0 \epsilon \delta \le \epsilon + \frac{1}{2} \epsilon^{\frac{3}{2}} \le 2\epsilon \, , \tag{55}$$

and combining this with (54) :

$$\|(T_1 A)^{df}(0)\|_{H^s} \le \epsilon + \epsilon^{\frac{3}{2}} \le 2\epsilon \, . \tag{56}$$

Similarly we also obtain by Lemma 4.1

$$\begin{aligned}
\|\partial_t (T_1 A)^{cf}(0)\|_{H^{s-1}} &\lesssim c_0 \epsilon \delta = c_0 \epsilon^2 \le \frac{1}{2} \epsilon^{\frac{3}{2}} \\
\|\partial_t (T_1 A)(0)\|_{H^{s-1}} &\le \epsilon + \frac{1}{2} \epsilon^{\frac{3}{2}} \\
\|\partial_t (T_1 A)^{df}(0)\|_{H^{s-1}} &\le \epsilon + \epsilon^{\frac{3}{2}} \le 2\epsilon \, ,
\end{aligned}$$

and

$$\|\partial_t (T_1 \phi)(0)\|_{H^s} + \|(\partial_t T_1 \phi)(0)\|_{H^{s-1}} \le \epsilon + \frac{1}{2} \epsilon^{\frac{3}{2}} \, .$$

We have now shown that (49) with ϵ replaced by $\epsilon + \frac{1}{2}\epsilon^{\frac{3}{2}}$ and (51) with $\delta = \frac{1}{2}\epsilon^{\frac{3}{2}}$ are fulfilled with A and ϕ replaced by $T_1 A$ and $T_1 \phi$.

In a next step we define $V_2 := -(-\Delta)^{-1} div (T_1 A)(0)$ so that $\nabla V_2 = (T_1 A)^{cf}(0)$ and thus by (54)

$$\|V_2\|_X = \|\nabla V_2\|_{H^s} \le \epsilon^{\frac{3}{2}} \, . \tag{57}$$

We define the next gauge transform T_2 by

$$\begin{aligned}
A &\longmapsto U_2 T_1 A U_2^{-1} - \nabla U_2 U_2^{-1} \\
\phi &\longmapsto U_2 T_1 \phi U_2^{-1}
\end{aligned}$$

with $U_2 = \exp V_2$.

Calculating as above we obtain

$$\begin{aligned}
(T_2 A)(0) = (T_1 A)^{df}(0) &+ (\exp V_2 (T_1 A)^{df}(0) \exp(-V_2) - (T_1 A)^{df}(0)) \\
&+ ((\exp V_2 \nabla V_2 - \nabla(\exp V_2)) \exp(-V_2))
\end{aligned}$$

where we used $\nabla V_2 = (T_1 A)^{cf}(0)$. This implies :

$$\begin{aligned}
&\|(T_2 A)^{cf}(0)\|_{H^s} \\
&\le c_2 (\| \exp V_2 (T_1 A)^{df}(0)(\exp(-V_2) - I)\|_{H^s} + \|(\exp V_2 - I)(T_1 A)^{df}(0)\|_{H^s} \\
&\quad + \|((\exp V_2 \nabla V_2 - \nabla(\exp V_2)) \exp(-V_2))\|_{H^s}) \, .
\end{aligned}$$

The first two terms on the right hand side are bounded by (57) by

$$c_2((\exp(c_1\|V_2\|_X)-1)+1)2\epsilon(\exp(c_1\|V_2\|_X)-1)\lesssim(\epsilon^{\frac{3}{2}}+1)\epsilon\epsilon^{\frac{3}{2}}\lesssim\epsilon^{\frac{5}{2}}\le\frac{1}{4}\epsilon^2\,,$$

where we used (56), whereas the last term on the right hand side can be handled similarly as in the first iteration step :

$$c_2\|((\exp V_2\nabla V_2-\nabla(\exp V_2))\exp(-V_2))\|_{H^s}\lesssim\epsilon^3\le\frac{1}{4}\epsilon^2\,.$$

This implies

$$\|(T_2A)^{cf}(0)\|_{H^s}\le\frac{1}{2}\epsilon^2$$

and also

$$\|(T_2A)(0)\|_{H^s}\le\|(T_1A)^{df}(0)\|_{H^s}+\frac{1}{2}\epsilon^2\le\epsilon+\epsilon^{\frac{3}{2}}+\frac{1}{2}\epsilon^2\le2\epsilon\,,$$

thus

$$\|(T_2A)^{df}(0)\|_{H^s}\le\epsilon+\epsilon^{\frac{3}{2}}+\epsilon^2\le2\epsilon\,.$$

Similar estimates are also obtained for $\|\partial_t(T_2A)^{cf}(0)\|_{H^{s-1}}$, $\|\partial_t(T_2A)(0)\|_{H^{s-1}}$ and $\|\partial_t(T_2A)^{df}(0)\|_{H^s}$ We also obtain

$$\|(T_2\phi)(0)\|_{H^s}+\|(\partial_tT_2\phi)(0)\|_{H^{s-1}}\le\epsilon+\epsilon^{\frac{3}{2}}+\epsilon^2\,.$$

We have now shown that (49) with ϵ replaced by $\epsilon+\epsilon^{\frac{3}{2}}+\epsilon^2$ and (51) with $\delta=\frac{1}{2}\epsilon^2$ are fulfilled with A and ϕ replaced by T_2A and $T_2\phi$.

By iteration we obtain a sequence of gauge transforms T_k defined by

$$A\longmapsto\prod_{l=k}^{1}\exp V_lA\prod_{l=1}^{k}\exp(-V_l)-\nabla(\prod_{l=k}^{1}\exp V_l)\prod_{l=1}^{k}\exp(-V_l)$$

$$\phi\longmapsto\prod_{l=k}^{1}\exp V_l\,\phi\prod_{l=1}^{k}\exp(-V_l)$$

with

$$V_l:=-(-\Delta)^{-1}div\,(T_{l-1}A)(0)$$

where $T_0:=id$. We remark that $\nabla V_k=(T_{k-1}A)^{cf}(0)$. We now make the assumption that for some $k\ge2$ we know that

$$\|(T_{k-1}A)^{df}(0)\|_{H^s}\le\epsilon+\epsilon^{\frac{3}{2}}+...+\epsilon^{\frac{k+1}{2}}\le2\epsilon$$

and

$$\|V_k\|_X=\|(T_{k-1}A)^{cf}(0)\|_{H^s}\le\frac{1}{2}\epsilon^{\frac{k+1}{2}}\,. \tag{58}$$

This holds for the case $k=2$ as shown before. Exactly as in the first two steps we obtain the estimate (with implicit constants independent of k from now on) :

$$\begin{aligned}
&\|V_{k+1}\|_X=\|\nabla V_{k+1}\|_{H^s}=\|(T_kA)^{cf}(0)\|_{H^s}\\
&\lesssim((\exp(c_1\|V_k\|_X)-1)+1)\|(T_{k-1}A)^{df}(0)\|_{H^s}(\exp(c_1\|V_k\|_X-1)\\
&\quad+(\exp(c_1\|V_k\|_X)-1)\|\nabla V_k\|_{H^s}+(\exp(c_1\|V_k\|_X)-1-c_1\|V_k\|_X)\\
&\lesssim(\|(T_{k-1}A)^{cf}(0)\|_{H^s}+1)\|(T_{k-1}A)^{df}(0)\|_{H^s}\|(T_{k-1}A)^{cf}(0)\|_{H^s}\\
&\quad+\|(T_{k-1}A)^{cf}(0)\|_{H^s}\|(T_{k-1}A)^{df}(0)\|_{H^s}+\|(T_{k-1}A)^{cf}(0)\|^2_{H^s}\\
&\lesssim(\epsilon^{\frac{k+1}{2}}+1)2\epsilon\epsilon^{\frac{k+1}{2}}+\epsilon^{\frac{k+1}{2}}\epsilon^{\frac{k+1}{2}}+\epsilon^{k+1}\lesssim\epsilon^{\frac{k+3}{2}}+\epsilon^{k+1}\le\frac{1}{2}\epsilon^{\frac{k}{2}+1}
\end{aligned}$$

and

$$\|(T_kA)(0)\|_{H^s}\le\|(T_{k-1}A)^{df}(0)\|_{H^s}+\frac{1}{2}\epsilon^{\frac{k}{2}+1}\le\epsilon+\epsilon^{\frac{3}{2}}+...+\epsilon^{\frac{k+1}{2}}+\frac{1}{2}\epsilon^{\frac{k}{2}+1}\le2\epsilon\,,$$

thus

$$\|(T_kA)^{df}(0)\|_{H^s} \le \epsilon + \epsilon^{\frac{3}{2}} + ... + \epsilon^{\frac{k}{2}+1} \le 2\epsilon\,. \tag{59}$$

Thus these estimates hold for any $k \ge 2$. Similarly one can show that

$$\begin{aligned}&\|(\partial_t T_k A)(0)\|_{H^{s-1}} + \|(\partial_t T_k A)^{df}(0)\|_{H^{s-1}} + \|(T_k\phi)(0)\|_{H^s} + \|(\partial_t T_k\phi)(0)\|_{H^{s-1}}\\ &\quad \lesssim \epsilon\,. \end{aligned}\tag{60}$$

Next we estimate

$$\begin{aligned}\|T_kA\|_{H^s} &\le \|(\prod_{l=k}^1 \exp V_l) A \prod_{l=1}^k \exp(-V_l)\|_{H^s} + \|\nabla(\prod_{l=k}^1 \exp V_l) \prod_{l=1}^k \exp(-V_l)\|_{H^s}\\ &= I + II\,.\end{aligned}$$

We further estimate

$$\begin{aligned} I &\le \|A\|_{H^s} + \|((\prod_{l=k}^1 \exp V_l) - I) A ((\prod_{l=1}^k \exp(-V_l) - I)\|_{H^s}\\ &\quad + \|A((\prod_{l=1}^k \exp(-V_l) - I)\|_{H^s} + \|((\prod_{l=k}^1 \exp V_l) - I)A\|_{H^s}\\ &= \|A\|_{H^s} + I_1 + I_2 + I_3 \end{aligned}$$

In order to control I_1 we consider first

$$\begin{aligned} \|\prod_{l=k}^1 \exp V_l - I\|_X &= \|\prod_{l=k}^1 \sum_{n=0}^\infty \frac{V_l^n}{n!} - I\|_X = \|\sum_{m=1}^\infty \sum_{n_1+...+n_k=m} \prod_{l=k}^1 \frac{V_l^{n_l}}{n_l!}\|_X\\ &\lesssim \sum_{m=1}^\infty \sum_{n_1+...+n_k=m} \prod_{l=k}^1 \frac{(c_1\|V_l\|_X)^{n_l}}{n_l!} = \prod_{l=k}^1 \exp(c_1\|V_l\|_X) - 1 \qquad (61)\\ &= \exp(\sum_{l=1}^k c_1\|V_l\|_X) - 1 \lesssim \exp(\sum_{l=1}^k c_1 \epsilon^{\frac{l+1}{2}}) - 1 \lesssim \exp(2c_1\epsilon) - 1 \lesssim \epsilon \end{aligned}$$

independently of k where we used (58). Consequently

$$I_1 \lesssim \|(\prod_{l=k}^1 \exp V_l) - I\|_X \|A\|_{H^s} \|(\prod_{l=1}^k \exp(-V_l) - I\|_X \lesssim \|A\|_{H^s}\epsilon^2\,.$$

Estimating I_2 and I_3 similarly we obtain

$$I \lesssim \|A\|_{H^s}(1 + \epsilon^2 + \epsilon)\,.$$

Moreover

$$\begin{aligned} II &\le \|\nabla(\prod_{l=k}^1 \exp V_l - I)(\prod_{l=1}^k \exp(-V_l) - I)\|_{H^s} + \|\nabla(\prod_{l=k}^1 \exp V_l - I)\|_{H^s}\\ &\lesssim \|\prod_{l=k}^1 \exp V_l - I\|_X \|(\prod_{l=1}^k \exp(-V_l) - I)\|_X + \|(\prod_{l=k}^1 \exp V_l - I)\|_X\\ &\lesssim \epsilon^2 + \epsilon\,, \end{aligned}$$

Summarizing we obtain with implicit constants which are independent of k :

$$\|T_kA\|_{H^s} \lesssim \|A\|_{H^s} + \epsilon$$

Similarly we also obtain

$$\|\partial_t(T_kA)\|_{H^{s-1}} \lesssim \|\partial_t A\|_{H^{s-1}}$$

and

$$\|T_k\phi\|_{H^s} \lesssim \|\phi\|_{H^s} \quad , \quad \|\partial_t(T_k\phi)\|_{H^{s-1}} \lesssim \|\partial_t\phi\|_{H^{s-1}}\,.$$

We want to consider the mapping T defined by $TA = \lim_{k\to\infty} T_k A$ and $T\phi = \lim_{k\to\infty} T_k\phi$, where the limit is taken in $C^0([0,1],H^s)\cap C^1([0,1],H^{s-1})$. This would imply by (58): $\|(TA)^{cf}(0)\|_{H^s} = \lim_{k\to\infty}\|(T_kA)^{cf}\|_{H^s} = 0$, thus the desired property

$$(TA)^{cf}(0) = 0\,.$$

Now define

$$SA := \prod_{l=\infty}^{1}(\exp V_l)A\prod_{l=1}^{\infty}\exp(-V_l) - \nabla(\prod_{l=\infty}^{1}\exp V_l)\prod_{1=1}^{\infty}\exp(-V_l) = UAU^{-1} - \nabla UU^{-1}\,,$$

with $U := \prod_{l=\infty}^{1}\exp V_l$, where the limit is taken with respect to $\|\cdot\|_X$.
This limit in fact exists, because by the calculations in (61) we obtain for $N>k$ the estimate

$$\|\prod_{l=N}^{1}\exp V_l - \prod_{l=k}^{1}\exp V_l\|_X \lesssim \|\prod_{l=N}^{k+1}\exp V_l - I\|_X(\|\prod_{l=k}^{1}\exp V_l - I\|_X + 1) \lesssim \epsilon^{\frac{k}{2}+1}(\epsilon+1)\,.$$

We also obtain $U^{-1} = \prod_{l=1}^{\infty}\exp(-V_l)$, which is defined in the same way.
In order to prove $S=T$ we estimate as follows :

$$\begin{aligned}
&\|SA - T_kA\|_{H^s}\\
&\le \|(\prod_{l=\infty}^{1}\exp V_l - \prod_{l=k}^{1}\exp V_l)A\prod_{l=1}^{\infty}\exp(-V_l)\|_{H^s}\\
&+ \|\prod_{l=k}^{1}\exp V_l A(\prod_{l=1}^{\infty}\exp(-V_l) - \prod_{l=1}^{k}\exp(-V_l))\|_{H^s}\\
&+ \|\nabla(\prod_{l=\infty}^{1}(\exp V_l) - \prod_{l=k}^{1}\exp V_l)(\prod_{l=1}^{\infty}\exp(-V_l)\|_{H^s}\\
&+ \|\nabla(\prod_{l=k}^{1}\exp V_l(\prod_{l=1}^{\infty}\exp(-V_l) - \prod_{l=1}^{k}\exp(-V_l))\|_{H^s}\\
&= I + II + III + IV
\end{aligned}$$

Now

$$\begin{aligned}
I &= \|(\prod_{l=\infty}^{k+1}\exp V_l - I)\prod_{l=k}^{1}\exp V_l A\prod_{l=1}^{\infty}\exp(-V_l)\|_{H^s}\\
&\lesssim \|\prod_{l=\infty}^{k+1}\exp V_l - I\|_X(\|\prod_{l=k}^{1}\exp V_l - I\|_X + 1)\|A\prod_{l=1}^{\infty}\exp(-V_l)\|_{H^s}\,.
\end{aligned}$$

Now by (61) we obtain

$$\|\prod_{l=k}^{1}\exp V_l - I\|_X \lesssim \epsilon\,,$$

and

$$\|A\prod_{l=1}^{\infty}\exp(-V_l)\|_{H^s} \lesssim \|A\|_{H^s}(1 + \|\prod_{l=1}^{\infty}\exp(-V_l) - I\|_X) \lesssim \|A\|_{H^s}(1+\epsilon)$$

and also similarly as in (61)

$$\|\prod_{l=\infty}^{k+1}\exp(-V_l) - I\|_X \le \exp(\sum_{l=k+1}^{\infty} c_1\epsilon^{\frac{k}{2}+1}) - 1 \lesssim \exp(c\epsilon^{\frac{k}{2}+1}) - 1 \lesssim \epsilon^{\frac{k}{2}+1}$$

so that

$$I + II \lesssim \epsilon^{\frac{k}{2}+1} \|A\|_{H^s} \,.$$

Next we estimate

$$\begin{aligned}
III & \le \|\nabla(\prod_{l=\infty}^1 (\exp V_l) - \prod_{l=k}^1 \exp V_l)((\prod_{l=1}^{\infty} \exp(-V_l) - I)\|_{H^s} \\
& \quad + \|\nabla(\prod_{l=\infty}^1 (\exp V_l) - \prod_{l=k}^1 \exp V_l)\|_{H^s} \\
& \lesssim \|I - \prod_{l=\infty}^{k+1} \exp V_l\|_X (\|\prod_{l=k}^1 \exp V_l - I\|_X + 1)(\|\prod_{l=1}^{\infty} \exp(-V_l) - I\|_X + 1) \\
& \lesssim \epsilon^{\frac{k}{2}+1}(\epsilon+1)(\epsilon+1) \lesssim \epsilon^{\frac{k}{2}+1} \,.
\end{aligned}$$

Finally

$$\begin{aligned}
IV & \le \|\nabla(\prod_{l=k}^1 \exp V_l - I) \prod_{l=1}^k \exp(-V_l)(I - \prod_{l=k+1}^{\infty} \exp(-V_l))\|_{H^s} \\
& \lesssim \|\prod_{l=k}^1 \exp V_l - I\|_X (\prod_{l=1}^k \exp(-V_l) - I\|_X + 1)\|I - \prod_{l=k+1}^{\infty} \exp(-V_l)\|_X \\
& \lesssim \epsilon(\epsilon+1)\epsilon^{\frac{k}{2}+1} \lesssim \epsilon^{\frac{k}{2}+2} \,,
\end{aligned}$$

so that we obtain

$$\|SA - T_k A\|_{H^s} \lesssim \epsilon^{\frac{k}{2}+1}(\|A\|_{H^s} + 1) \,\to\, 0 \quad (k \to \infty) \,,$$

thus $T_k A \to SA$ in $C^0([0,1],H^s)$ and similarly $\partial_t T_k A \to \partial_t SA$ in $C^0([0,1],H^{s-1})$ as well as $T_k\phi \to S\phi$ in $C^0([0,1],H^s)$ and $\partial_t T_k \phi \to \partial_t S\phi$ in $C^0([0,1],H^{s-1})$. We have shown that $T = S$ is a gauge transformation which besides fulfilling the temporal gauge has the property $(TA)^{cf}(0) = 0$ and preserves the regularity $A, \phi \in C^0([0,1],H^s) \cap C^1([0,1],H^{s-1})$. From the properties (59) and (60) of T_k we also deduce

$$\|(TA)^{df}(0)\|_{H^s} + \|(\partial_t TA)^{df}(0)\|_{H^{s-1}} + \|(T\phi)(0)\|_{H^s} + \|(\partial_t T\phi)(0)\|_{H^{s-1}} \lesssim \epsilon \,.$$

Assume now that $A = A_- + A_+ + A'$, where $A_\pm \in X_\pm^{s,\frac{3}{4}+}[0,1]$, $A' \in X_{\tau=0}^{s+\alpha,\frac{1}{2}+}[0,1]$ and $\partial_t A' \in C^0([0,1],H^{s-1})$. Let

$$TA = UAU^{-1} - \nabla U U^{-1} \,,$$

where $U = \prod_{l=\infty}^1 \exp V_l$, is defined as above. We want to show that TA has the same regularity. Let $\psi = \psi(t)$ be a smooth function with $\psi(t) = 1$ for $0 \le t \le 1$ and $\psi(t) = 0$ for $t \ge 2$. Then we obtain by Lemma 4.3 below and (61) :

$$\begin{aligned}
\|UA_\pm \psi\|_{X_\pm^{s,\frac{3}{4}+}} & \lesssim \|\nabla U \psi\|_{X_\pm^{s,1}} \|A_\pm\|_{X_\pm^{s,\frac{3}{4}+}} \lesssim \|\nabla U\|_{H^s} \|A_\pm\|_{X_\pm^{s,\frac{3}{4}+}} \\
& \lesssim \|U - I\|_X \|A_\pm\|_{X_\pm^{s,\frac{3}{4}+}} \lesssim \epsilon \|A_\pm\|_{X_\pm^{s,\frac{3}{4}+}} \,,
\end{aligned}$$

thus

$$\|UA_\pm\|_{X_\pm^{s,\frac{3}{4}+}[0,1]} \lesssim \epsilon \|A_\pm\|_{X_\pm^{s,\frac{3}{4}+}[0,1]} \,.$$

Similarly we obtain

$$\|UA_\pm U^{-1}\|_{X_\pm^{s,\frac{3}{4}+}[0,1]} \lesssim \epsilon \|UA_\pm\|_{X_\pm^{s,\frac{3}{4}+}} \lesssim \epsilon^2 \|A_\pm\|_{X_\pm^{s,\frac{3}{4}+}[0,1]} < \infty \,.$$

We also have

$$\|(\nabla U)\psi U^{-1}\psi\|_{X_{\pm}^{s,\frac{3}{4}+}} \lesssim \|\nabla U\psi\|_{X_{\pm}^{s,\frac{3}{4}+}} \|\nabla(U^{-1})\psi\|_{X_{\pm}^{s,1}} \lesssim \|\nabla U\|_{H^s} \|\nabla(U^{-1})\|_{H^s} \,,$$

thus

$$\|(\nabla U)U^{-1}\|_{X_{\pm}^{s,\frac{3}{4}+}[0,1]} \lesssim \|\nabla U\|_{H^s} \|\nabla(U^{-1})\|_{H^s} \,.$$

Moreover by Sobolev we obtain

$$\begin{aligned}\|UA'\psi\|_{X_{\tau=0}^{s+\alpha,\frac{1}{2}+}} &\lesssim \|\nabla(U)\psi\|_{X_{\tau=0}^{s,1}} \|A'\|_{X_{\tau=0}^{s+\alpha,\frac{1}{2}+}} \\ &\lesssim \|\nabla U\|_{H^s} \|A'\|_{X_{\tau=0}^{s+\alpha,\frac{1}{2}+}} \lesssim \epsilon \|A'\|_{X_{\tau=0}^{s+\alpha,\frac{1}{2}+}} \,.\end{aligned}$$

Similarly as before this implies

$$\|UA'U^{-1}\|_{X_{\tau=0}^{s+\alpha,\frac{1}{2}+}[0,1]} \lesssim \epsilon^2 \|A'\|_{X_{\tau=0}^{s+\alpha,\frac{1}{2}+}[0,1]} < \infty \,.$$

By Sobolev's muliplication law we also obtain

$$\|U\partial_t A'\|_{C^0([0,1],H^{s-1})} \lesssim \|\nabla U\|_{H^s} \|\partial_t A'\|_{C^0([0,1],H^{s-1})} \lesssim \epsilon \|\partial_t A'\|_{C^0([0,1],H^{s-1})} \,.$$

As before this implies

$$\|U\partial_t A'U^{-1}\|_{C^0([0,1],H^{s-1})} \lesssim \epsilon^2 \|\partial_t A'\|_{C^0([0,1],H^{s-1})} < \infty \,.$$

We have thus shown that TA has the same regularity as A. The same estimates also show that

$$\|U\phi_{\pm}U^{-1}\|_{X_{\pm}^{s,\frac{3}{4}+}[0,1]} \lesssim \epsilon^2 \|\phi_{\pm}\|_{X_{\pm}^{s,\frac{3}{4}+}[0,1]} < \infty \,,$$

so that $T\phi = U\phi U^{-1}$ maps $X_{+}^{s+\frac{1}{4},\frac{1}{2}+} + X_{-}^{s+\frac{1}{4},\frac{1}{2}+}$ into itself. The same properties also hold for its inverse T^{-1} which is given by

$$\begin{aligned} B &\longmapsto U^{-1}BU + U^{-1}\nabla U \\ \phi' &\longmapsto U\phi' U^{-1} \,. \end{aligned}$$

□

In the last proof we used the following

Lemma 4.3. *The following estimate holds for $s > \frac{n}{2} - \frac{3}{4}$ and $\epsilon > 0$ sufficiently small:*

$$\|uv\|_{X_{\pm}^{s,\frac{3}{4}+\epsilon}} \lesssim \|\nabla u\|_{X_{\pm}^{s,1}} \|v\|_{X_{\pm}^{s,\frac{3}{4}+\epsilon}} \,.$$

Proof. By Tao [T], Cor. 8.2 we may replace ∇ by $\langle \nabla \rangle$ so that it suffices to prove

$$\|uv\|_{X_{\pm}^{s,\frac{3}{4}+\epsilon}} \lesssim \|u\|_{X_{\pm}^{s+1,1}} \|v\|_{X_{\pm}^{s,\frac{3}{4}+\epsilon}} \,.$$

We start with the elementary estimate

$$|(\tau_1+\tau_2) \mp |\xi_1+\xi_2|| \le |\tau_1 \mp |\xi_1|| + |\tau_2 \mp |\xi_2|| + |\xi_1| + |\xi_2| - |\xi_1+\xi_2| \,.$$

Assume now w.l.o.g. $|\xi_2| \ge |\xi_1|$. We have

$$|\xi_1| + |\xi_2| - |\xi_1+\xi_2| \le |\xi_1| + |\xi_2| + |\xi_1| - |\xi_2| = 2|\xi_1| \,,$$

so that

$$|(\tau_1+\tau_2) \mp |\xi_1+\xi_2|| \le |\tau_1 \mp \xi_1| + |\tau_2 \mp |\xi_2|| + 2\min(|\xi_1|,|\xi_2|) \,.$$

Using Fourier transforms by standard arguments it thus suffices to show the following three estimates:

$$\|uv\|_{X^{s,0}_{\pm}} \lesssim \|u\|_{X^{s+1,\frac{1}{4}-\epsilon}_{\pm}} \|v\|_{X^{s,\frac{3}{4}+\epsilon}_{\pm}}$$
$$\|uv\|_{X^{s,0}_{\pm}} \lesssim \|u\|_{X^{s+1,1}_{\pm}} \|v\|_{X^{s,0}_{\pm}}$$
$$\|uv\|_{X^{s,0}_{\pm}} \lesssim \|u\|_{X^{s+\frac{1}{4}-\epsilon,1}_{\pm}} \|v\|_{X^{s,\frac{3}{4}+\epsilon}_{\pm}}$$

The first and second estimate easily follow from Sobolev, whereas the last one is implied by [FK] , Thm. 1.1. □

5. Proof of Theorem 1.2 and Theorem 1.1

Proof. We only prove Theorem 1.2. It suffices to construct a unique local solution of (6),(7),(8) with initial conditions

$$A^{df}(0) = a^{df} \, , \, (\partial_t A^{df})(0) = a'^{df} \, , \, A^{cf}(0) = a^{cf} \, , \, \phi(0) = \phi_0 \, , \, (\partial_t \phi)(0) = \phi_1 \, ,$$

which fulfill

$$\|A^{df}(0)\|_{H^s} + \|(\partial_t A)^{df}(0)\|_{H^{s-1}} + \|A^{cf}(0)\|_{H^s} + \|\phi(0)\|_{H^s} + \|(\partial_t \phi)(0)\|_{H^{s-1}} \le \epsilon$$

for a sufficiently small $\epsilon > 0$. By Lemma 4.1 there exists a gauge transformation T which fulfills (50) and $(TA)^{cf}(0) = 0$. We use Prop. 3.2 to construct a unique solution $(\tilde{A}, \tilde{\phi})$ of (6),(7),(8) , where $\tilde{A} = \tilde{A}^{df}_+ + \tilde{A}^{df}_- + \tilde{A}^{cf}$ and $\tilde{\phi} = \tilde{\phi}_+ + \tilde{\phi}_-$, with data

$$\tilde{A}^{df}(0) = (TA)^{df}(0) \, , \, (\partial_t \tilde{A})^{df}(0) = (\partial_t (TA)^{df})(0) \, , \, \tilde{A}^{cf}(0) = (TA)^{cf}(0) = 0 \, ,$$
$$\tilde{\phi}(0) = (T\phi)(0) \, , \, (\partial_t \tilde{\phi})(0) = (\partial_t T\phi)(0)$$

with the regularity

$$\tilde{A}^{df}_{\pm} \in X^{s,\frac{3}{4}+}_{\pm}[0,1], \tilde{A}^{cf} \in X^{s+\alpha,\frac{1}{2}+}_{\tau=0}[0,1], \partial_t \tilde{A}^{cf} \in C^0([0,1],H^{s-1}), \tilde{\phi}_{\pm} \in X^{s,\frac{3}{4}+}_{\pm}[0,1] \, .$$

This solution satisfies also $\tilde{A}, \tilde{\phi} \in C^0([0,1],H^s) \cap C^1[0,1],H^{s-1})$.

Applying the inverse gauge transformation T^{-1} according to Lemma 4.1 we obtain a unique solution of (6),(7),(8) with the required initial data and also the same regularity.

The proof of Theorem 1.1 is completely analogous by use of Prop. 3.1. □

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